

# A note on convergent isocrystals on simply connected varieties

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## Abstract

It is conjectured by de Jong that, if  $X$  is a connected projective smooth variety over an algebraically closed field  $k$  of characteristic  $p > 0$  with trivial étale fundamental group, any convergent isocrystal  $\mathcal{E}$  on  $X$  is trivial. We discuss this conjecture when  $X$  is liftable to characteristic zero, and prove the triviality of  $\mathcal{E}$  in this case under certain conditions on (semi)stability.

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## Introduction

For a connected quasi-projective smooth variety  $X$  over an algebraically closed field  $k$  of characteristic zero, it is proved by Malcev [12] and Grothendieck [8] that, if the étale fundamental group  $\pi_1^{\text{ét}}(X)$  is trivial, any coherent  $\mathcal{O}_X$ -module with integrable connection on  $X$  over  $k$  (which is equivalent to an  $\mathcal{O}_X$ -coherent  $\mathcal{D}_X$ -module on  $X$ ) is trivial (isomorphic to a direct sum of  $(\mathcal{O}_X, d)$ ). The latter assertion is equivalent to the triviality of the de Rham fundamental group  $\pi_1^{\text{dR}}(X)$  defined as the Tannaka dual of the category of coherent  $\mathcal{O}_X$ -modules with integrable connection on  $X$  over  $k$ . So the above theorem can be interpreted as an interesting relation between  $\pi_1^{\text{ét}}(X)$  and  $\pi_1^{\text{dR}}(X)$ .

As an analogue of this theorem in characteristic  $p > 0$ , Esnault and Mehta [5], [6] proved a conjecture of Gieseker which says, for a connected projective smooth variety  $X$  over an algebraically closed field  $k$  of characteristic  $p > 0$  with trivial

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etale fundamental group, any stratified bundle on  $X$  (which is equivalent to an  $\mathcal{O}_X$ -coherent  $\mathcal{D}_X$ -module on  $X$ ) is trivial. (In [7], the same statement for connected quasi-projective smooth  $X$  is also proven under certain assumption.) The latter assertion is equivalent to the triviality of the stratified fundamental group  $\pi_1^{\text{strat}}(X)$  defined as the Tannaka dual of the category of stratified bundles on  $X$ , and the theorem reveals an interesting relation between  $\pi_1^{\text{et}}(X)$  and  $\pi_1^{\text{strat}}(X)$ .

As a  $p$ -adic version of the above theorem, there is the following conjecture, which is raised by de Jong according to private communication of the author with Esnault:

**Conjecture 0.1** (de Jong). *For a connected projective smooth variety  $X$  over an algebraically closed field  $k$  of characteristic  $p > 0$  with trivial étale fundamental group, any convergent isocrystal on  $X$  over  $K$  (where  $K$  is the fraction field of a complete discrete valuation ring  $V$  of mixed characteristic with residue field  $k$ ) is trivial.*

As in the previous cases, the conclusion of the conjecture is equivalent to the triviality of the convergent fundamental group  $\pi_1^{\text{conv}}(X)$  defined as the Tannaka dual of the category of convergent isocrystals on  $X$ .

In this paper, we discuss this conjecture when  $X$  is liftable to a projective smooth scheme  $\mathfrak{X}$  over  $\mathrm{Spec} V$  (and  $V$  admits a lift of absolute Frobenius on  $k$ ). When the étale fundamental group  $\pi_1^{\mathrm{ét}}(\overline{\mathbf{X}})$  of the generic geometric fiber  $\overline{\mathbf{X}}$  of  $\mathfrak{X}$  is trivial, the conjecture in this case easily follows from the aforementioned result of Malcev and Grothendieck. But this does not imply the conjecture in general liftable case because we do not know in general whether  $\pi_1^{\mathrm{ét}}(\overline{\mathbf{X}})$  is trivial or not (although we know the triviality of its prime-to- $p$  quotients).

We prove theorems (Theorem 1.7 and 1.9) which roughly claim that, when  $\mathcal{E}$  admits very nice ‘mod  $\varpi$  reductions’ (where  $\varpi$  is a uniformizer of  $V$ ) which are  $\mu$ -stable or semistable sheaves on  $X$ ,  $\mathcal{E}$  is trivial. As corollaries, we prove the triviality of  $\mathcal{E}$  when  $\mathcal{E}$  is of rank 1 (Corollary 1.8) or when  $\mathcal{E}$  admits a structure of a convergent  $F$ -isocrystal with ‘strongly semistable mod  $\varpi$  reduction’ (Corollary 1.10, 1.11).

The rough idea of the proof is the following: First we prove the triviality of ‘ $\mathcal{E}$  modulo  $\varpi$ ’ using the result of Esnault-Mehta [5] and the moduli of stable sheaves on  $X$  constructed by Adrian Langer [1], [2]. Then, we prove the triviality of ‘ $\mathcal{E}$  modulo  $\varpi^N$ ’ ( $N \in \mathbb{N}$ ), which is a deformation of ‘ $\mathcal{E}$  modulo  $\varpi$ ’, by checking the behavior of deformation class under the level raising Frobenius pullback functor of Berthelot [4]. Finally, we prove the triviality of  $\mathcal{E}$  by using Langton’s theorem [11] or the moduli of semistable sheaves on  $\mathfrak{X}$  constructed by Langer.

The author is grateful to H  l  ne Esnault, because he started this study after the private communication with her on the above conjecture of de Jong. He is partly supported by JSPS Grant-in-Aid for Scientific Research (C) 25400008 and Grant-in-Aid for Scientific Research (B) 23340001.

# 1 Preliminaries and statement of results

Throughout this paper, let  $p$  be a fixed prime number,  $k$  an algebraically closed field of characteristic  $p$ ,  $V$  a complete discrete valuation ring of mixed characteristic with residue field  $k$  and  $K$  the fraction field of  $V$ . Let  $\varpi$  be a uniformizer of  $V$ . Denote the absolute ramification index of  $V$  by  $e$  and let  $\mathfrak{e}$  be the minimal natural number with  $e \leq p^{\mathfrak{e}}(p-1)$ . We denote the absolute Frobenius morphism on  $\mathrm{Spec} k$  by  $\sigma$ . We assume that there exists a lift  $\sigma_V : \mathrm{Spec} V \rightarrow \mathrm{Spec} V$  of  $\sigma$  and fix it. We denote the  $p$ -adic completion  $\mathrm{Spf} V \rightarrow \mathrm{Spf} V$  of  $\sigma_V$  by the same symbol. Denote the morphism  $\mathrm{Spec} K \rightarrow \mathrm{Spec} K$  induced by  $\sigma_V$  by  $\sigma_K$ . Note that, when  $V$  is the Witt ring  $W(k)$  of  $k$ , the condition on the lift of Frobenius is satisfied and  $\mathfrak{e} = 0$ .

Let  $X$  be a projective smooth variety of dimension  $d$  over  $\mathrm{Spec} k$  and let  $\mathcal{X}$  be a projective smooth formal scheme over  $\mathrm{Spf} V$  with  $\mathcal{X} \otimes_V k = X$ . (Note that, throughout this paper, we assume the existence of such  $\mathcal{X}$ .) By GFGA,  $\mathcal{X}$  naturally corresponds to a projective smooth scheme over  $\mathrm{Spec} V$ , which we denote by  $\mathfrak{X}$ . We denote its generic fiber by  $\mathbf{X}$  and its geometric generic fiber by  $\overline{\mathbf{X}}$ . We assume that  $X$  and  $\overline{\mathbf{X}}$  are connected.

We denote the absolute Frobenius  $X \rightarrow X$  of  $X$  by  $F_{\mathrm{abs}}$ . For  $n \in \mathbb{N}$ , denote  $X \times_{\mathrm{Spec} k, \sigma^n} \mathrm{Spec} k$  (resp.  $\mathfrak{X} \times_{\mathrm{Spec} V, \sigma_V^n} \mathrm{Spec} V$ ) by  $X^{[n]}$  (resp.  $\mathfrak{X}^{[n]}$ ) and the  $p$ -adic completion of  $\mathfrak{X}^{[n]}$  by  $\mathcal{X}^{[n]}$ .  $X^{[n]}$  (resp.  $\mathfrak{X}^{[n]}$ ,  $\mathcal{X}^{[n]}$ ) is isomorphic to  $X$  (resp.  $\mathfrak{X}$ ,  $\mathcal{X}$ ) as schemes (resp. schemes, formal schemes), but we prefer to use this notation to avoid confusion.

Denote the projection  $X^{[i+1]} \rightarrow X^{[i]}$  ( $i \in \mathbb{N}$ ) by  $\pi$  and denote the  $n$ -times iteration of projections  $X^{[i+n]} \rightarrow X^{[i]}$  ( $i \in \mathbb{N}$ ) by  $\pi^n$ . Also, denote the relative Frobenius morphism  $X^{[i]} \rightarrow X^{[i+1]}$  ( $i \in \mathbb{N}$ ) by  $F$  and denote the  $n$ -times iteration of the relative Frobenius morphisms  $X^{[i]} \rightarrow X^{[i+n]}$  ( $i \in \mathbb{N}$ ) by  $F^n$ . Then  $F^n \circ \pi^n = F_{\mathrm{abs}}^n$  and  $F$  and  $\pi$  are ‘commutative’ in suitable sense as long as they are defined. Denote also the projection  $\mathfrak{X}^{[i+1]} \rightarrow \mathfrak{X}^{[i]}$  ( $i \in \mathbb{N}$ ) by  $\pi$  and denote the  $n$ -times iteration of projections  $\mathfrak{X}^{[i+n]} \rightarrow \mathfrak{X}^{[i]}$  ( $i \in \mathbb{N}$ ) by  $\pi^n$ . However, we do not assume the existence of a lift of a relative or absolute Frobenius morphism on  $X^{[i]}$  to that on  $\mathfrak{X}^{[i]}$ .

We fix an ample line bundle  $\mathcal{O}_{\mathfrak{X}}(1)$  on  $\mathfrak{X}$ . By restriction, this induces an ample line bundle on  $X, \mathbf{X}$ , which we denote by  $\mathcal{O}_X(1), \mathcal{O}_{\mathbf{X}}(1)$ , respectively. Also, by the pullback by  $\pi^n$ , this induces an ample line bundle on  $X^{[n]}, \mathbf{X}^{[n]}$ , which we denote by  $\mathcal{O}_{X^{[n]}}(1), \mathcal{O}_{\mathbf{X}^{[n]}}(1)$ , respectively. When we consider the slope  $\mu(\mathcal{F})$  or the reduced Hilbert polynomial  $p_{\mathcal{F}}$  of a torsion free sheaf  $\mathcal{F}$  on  $X^{[n]}$  (resp.  $\mathbf{X}^{[n]}$ ), we always consider them with respect to  $\mathcal{O}_{X^{[n]}}(1)$  (resp.  $\mathcal{O}_{\mathbf{X}^{[n]}}(1)$ ). (For the definition of  $\mu(\mathcal{F})$  and  $p_{\mathcal{F}}$ , see [9, Definition 1.2.11, 1.2.3].) Therefore, when we consider (semi)stability (also known as Gieseker (semi)stability) and  $\mu$ -(semi)stability, we consider them with respect to  $\mathcal{O}_{X^{[n]}}(1)$  (resp.  $\mathcal{O}_{\mathbf{X}^{[n]}}(1)$ ).

We give a review of rings of  $p$ -adic differential operators (arithmetic  $D$ -modules) defined by Berthelot [3], in our setting. Let  $\mathcal{X}$  be as above and let  $\mathcal{D}(1)_{(m)}$  (resp.  $\mathcal{D}(2)_{(m)}$ ) be the  $p$ -adically completed  $m$ -PD-envelope of  $\mathcal{X}$  in  $\mathcal{X} \times_V \mathcal{X}$  (resp.  $\mathcal{X} \times_V \mathcal{X} \times_V \mathcal{X}$ ). Then  $\mathcal{I} := \mathrm{Ker}(\mathcal{O}_{\mathcal{D}(1)_{(m)}} \rightarrow \mathcal{O}_{\mathcal{X}})$  is endowed with an  $m$ -PD-structure

and so we can define the ideal  $\mathcal{I}^{\{n\}}$  ( $n \in \mathbb{N}$ ) as in [3, 1.3.7]. We denote by  $\mathcal{D}(1)_{(m)}^n$  the closed formal subscheme of  $\mathcal{D}(1)_{(m)}$  defined by  $\mathcal{I}^{\{n\}}$ . Then we define  $\widehat{\mathcal{D}}_{\mathcal{X}}^{(m)}$  by  $\mathcal{D}_{\mathcal{X}/p^N, n}^{(m)} := \text{Hom}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{O}_{\mathcal{D}(1)_{(m)}}/p^N \mathcal{O}_{\mathcal{D}(1)_{(m)}}, \mathcal{O}_{\mathcal{X}}/p^N \mathcal{O}_{\mathcal{X}})$ ,  $\mathcal{D}_{\mathcal{X}/p^N}^{(m)} := \bigcup_{n \in \mathbb{N}} \mathcal{D}_{\mathcal{X}/p^N, n}^{(m)}$  and  $\widehat{\mathcal{D}}_{\mathcal{X}}^{(m)} := \varprojlim_{N \in \mathbb{N}} \mathcal{D}_{\mathcal{X}/p^N}^{(m)}$ . The morphism

$$\mathcal{D}(1)_{(m)} \times_{\mathcal{X}} \mathcal{D}(1)_{(m)} \cong \mathcal{D}(2)_{(m)} \longrightarrow \mathcal{D}(1)_{(m)}$$

(the isomorphism follows from the explicit description of  $m$ -PD envelope given in [3, 1.5]) induced by the projection  $\mathcal{X} \times_V \mathcal{X} \times_V \mathcal{X} \longrightarrow \mathcal{X} \times_V \mathcal{X}$  to the first and the third factors naturally defines the  $\mathcal{O}_{\mathcal{X}}$ -algebra structure on  $\widehat{\mathcal{D}}_{\mathcal{X}}^{(m)}$ . We put  $\widehat{\mathcal{D}}_{\mathcal{X}, \mathbb{Q}}^{(m)} := \mathbb{Q} \otimes_{\mathbb{Z}} \widehat{\mathcal{D}}_{\mathcal{X}}^{(m)}$  and finally we define  $\mathcal{D}_{\mathcal{X}, \mathbb{Q}}^{\dagger}$  by  $\mathcal{D}_{\mathcal{X}, \mathbb{Q}}^{\dagger} = \varinjlim_m \widehat{\mathcal{D}}_{\mathcal{X}, \mathbb{Q}}^{(m)}$ . Note that the above construction works also when  $\mathcal{X}$  is replaced by  $\mathcal{X}^{[i]}$  ( $i \in \mathbb{N}$ ).

Let  $\mathcal{E}$  be a convergent isocrystal on  $X/K$ . In terms of arithmetic  $D$ -modules, it is nothing but an  $\mathcal{O}_{\mathcal{X}, \mathbb{Q}}$ -coherent  $\mathcal{D}_{\mathcal{X}, \mathbb{Q}}^{\dagger}$ -module  $\mathcal{E}$ , where  $\mathcal{O}_{\mathcal{X}, \mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{X}}$  ([3, 4.1.4]). Let us denote the coherent  $\mathcal{O}_{\mathbf{X}}$ -module corresponding to  $\mathcal{E}$  via GFGA by  $\mathbf{E}$ . (Then  $\mathbf{E}$  is known to be locally free.) For each  $m \in \mathbb{N}$ ,  $\mathcal{E}$  naturally has a structure of  $\mathcal{O}_{\mathcal{X}, \mathbb{Q}}$ -coherent quasi-nilpotent  $\widehat{\mathcal{D}}_{\mathcal{X}, \mathbb{Q}}^{(m)}$ -module. The next proposition assures the existence of a certain  $\mathcal{O}_{\mathcal{X}}$ -coherent lattice of  $\mathcal{E}$ :

**Proposition 1.1.** *Let the notations be as above. Then, there exists a  $p$ -torsion free  $\mathcal{O}_{\mathcal{X}}$ -coherent quasi-nilpotent  $\widehat{\mathcal{D}}_{\mathcal{X}}^{(m)}$ -module  $\mathcal{E}^{(m)}$  with  $\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{E}^{(m)} = \mathcal{E}$  as  $\widehat{\mathcal{D}}_{\mathcal{X}, \mathbb{Q}}^{(m)}$ -modules.*

To prove it, we recall the notion of  $m$ -HPD-stratification in our setting.

**Definition 1.2.** *Let  $\mathcal{X}, \mathcal{D}(i)_{(m)}$  ( $i = 1, 2$ ) be as above. Also, let  $p_i : \mathcal{D}(1)_{(m)} \longrightarrow \mathcal{X}$  ( $i = 1, 2$ ),  $p_{ij} : \mathcal{D}(2)_{(m)} \longrightarrow \mathcal{D}(1)_{(m)}$  ( $1 \leq i < j \leq 3$ ) be the projections and let  $\Delta : \mathcal{X} \longrightarrow \mathcal{D}(1)_{(m)}$  be the diagonal map. Then we define an  $m$ -HPD-stratification on a coherent  $\mathcal{O}_{\mathcal{X}}$ -module or a coherent  $\mathcal{O}_{\mathcal{X}, \mathbb{Q}}$ -module  $\mathcal{E}$  as an  $\mathcal{O}_{\mathcal{D}(1)_{(m)}}$ -linear isomorphism  $\epsilon : p_2^* \mathcal{E} \longrightarrow p_1^* \mathcal{E}$  satisfying  $\Delta^*(\epsilon) = \text{id}$  and  $p_{12}^*(\epsilon) \circ p_{23}^*(\epsilon) = p_{13}^*(\epsilon)$ .*

Then, it is known (follows easily from [3, 2.3.7]) that, for a coherent  $\mathcal{O}_{\mathcal{X}}$ -module (resp. a coherent  $\mathcal{O}_{\mathcal{X}, \mathbb{Q}}$ -module)  $\mathcal{E}$ , giving a structure of quasi-nilpotent  $\widehat{\mathcal{D}}_{\mathcal{X}}^{(m)}$ -module on  $\mathcal{E}$  (resp. a quasi-nilpotent  $\widehat{\mathcal{D}}_{\mathcal{X}, \mathbb{Q}}^{(m)}$ -module on  $\mathcal{E}$ ) is equivalent to giving a structure of  $m$ -HPD-stratification on  $\mathcal{E}$ . Therefore, to prove Proposition 1.1, it suffices to prove the following:

**Lemma 1.3.** *Let  $\mathcal{E}$  be a coherent  $\mathcal{O}_{\mathcal{X}, \mathbb{Q}}$ -module endowed with an  $m$ -HPD-stratification  $\epsilon : p_2^* \mathcal{E} \longrightarrow p_1^* \mathcal{E}$ . Then there exists a  $p$ -torsion free coherent  $\mathcal{O}_{\mathcal{X}}$ -module  $\mathcal{F}$  with  $\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{F} = \mathcal{E}$  such that  $\epsilon_{\mathcal{F}} := \epsilon|_{p_2^* \mathcal{F}}$  induces the HPD-stratification  $p_2^* \mathcal{F} \longrightarrow p_1^* \mathcal{F}$  on  $\mathcal{F}$ .*

*Proof.* The proof is the level  $m$  version of [15, (0.7.4)], which is based on the technique of the proof of rigid analytic faithfully flat descent due to O. Gabber (cf.

[15, (0.7.2)], [14, (1.9)]). In this proof, we denote the  $p$ -adically completed tensor product of modules by  $\hat{\otimes}$  and use the symbol  $\otimes$  only for usual tensor products.

Take any  $p$ -torsion free coherent  $\mathcal{O}_{\mathcal{X}}$ -module  $\mathcal{F}'$  with  $\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{F}' = \mathcal{E}$ , and we define the map  $\theta : \mathcal{E} \rightarrow p_1^* \mathcal{E}$  by  $\theta(x) = \epsilon(p_2^* x)$ . By the description of  $m$ -PD envelope given in [3, 1.5], the maps  $p_i : \mathcal{D}(1)_{(m)} \rightarrow \mathcal{X}$  are flat. Hence  $p_1^* \mathcal{F}' \subset p_1^* \mathcal{E}$ . Put  $\phi' := \theta|_{\mathcal{F}'}$  and let  $\mathcal{F}$  be  $\theta^{-1}(p_1^* \mathcal{F}')$ .

First check the inclusion  $\mathcal{F} \subseteq \mathcal{F}'$ . Let  $\Delta_{\mathcal{E}} : p_1^* \mathcal{E} \rightarrow \mathcal{E}$  be the map defined by  $x \otimes \gamma \mapsto x \Delta^*(\gamma)$  for  $x \in \mathcal{E}$  and  $\gamma \in \mathcal{O}_{\mathcal{D}(1)_{(m)}}$ . Then  $\Delta_{\mathcal{E}} \circ \theta = \text{id}$  and  $\Delta_{\mathcal{E}}$  sends  $p_1^* \mathcal{F}'$  into  $\mathcal{F}'$ . If  $x$  is a local section of  $\mathcal{F}$ ,  $\theta(x) \in p_1^* \mathcal{F}'$  and so  $x = \Delta_{\mathcal{E}} \circ \theta(x) \in \mathcal{F}'$ . Hence  $\mathcal{F} \subseteq \mathcal{F}'$ .

Next we prove the coherence of  $\mathcal{F}$ . For an affine open formal subscheme  $\mathcal{U} = \text{Spf } A \subseteq \mathcal{X}$ , we denote  $\Gamma(\mathcal{U}, \mathcal{F}), \Gamma(\mathcal{U}, \mathcal{F}'), \Gamma(\mathcal{U}, \mathcal{E})$  by  $F_A, F'_A, E_A$ , respectively. Then, for any such  $A$ ,  $F'_A$  is a finitely generated module over the Noetherian ring  $A$  and  $F_A \subseteq F'_A$ . So  $F_A$  is a finitely generated  $A$ -module. Next, let us take affine open formal subschemes  $\mathcal{U}' = \text{Spf } A' \subseteq \mathcal{U} = \text{Spf } A \subseteq \mathcal{X}$ , and denote  $\mathcal{U} \times_{\mathcal{X}} \mathcal{D}(i)_{(m)}$  (resp.  $\mathcal{U}' \times_{\mathcal{X}} \mathcal{D}(i)_{(m)}$ ), which is an affine open formal subscheme of  $\mathcal{D}(i)_{(m)}$ , by  $\text{Spf } B(i)$  (resp.  $\text{Spf } B'(i)$ ). Then we have

$$\begin{aligned}
F_{A'} &= \theta_{A'}^{-1}(F'_{A'} \otimes_{A'} (B(1) \hat{\otimes}_A A')) \\
&= \text{Ker}(E_{A'} \xrightarrow{\theta_{A'}} E_{A'} \otimes_{A'} (B(1) \hat{\otimes}_A A') \rightarrow (E_{A'}/F'_{A'}) \otimes_{A'} (B(1) \hat{\otimes}_A A')) \\
&= \text{Ker}(E_{A'} \xrightarrow{\theta_{A'}} E_{A'} \otimes_{A'} (B(1) \hat{\otimes}_A A') \rightarrow (E_{A'}/F'_{A'}) \otimes_{A'} (B(1) \otimes_A A')) \\
&\hspace{25em} (E_{A'}/F'_{A'} \text{ is } p\text{-torsion}) \\
&= \text{Ker}(E_A \otimes_A A' \xrightarrow{\theta_A \otimes_A A'} E_A \otimes_A B(1) \otimes_A A' \rightarrow (E_A/F'_A) \otimes_A B(1) \otimes_A A') \\
&= \text{Ker}(E_A \xrightarrow{\theta_A} E_A \otimes_A B(1) \rightarrow (E_A/F'_A) \otimes_A B(1)) \otimes_A A' \\
&= \theta_A^{-1}(F'_A \otimes_A B(1)) \otimes_A A' = F_A \otimes_A A'.
\end{aligned}$$

So  $\mathcal{F}$  is coherent, as desired.

Next we prove that  $\theta(\mathcal{F})$  is contained in  $p_1^* \mathcal{F}$ . Since  $p_1$  is flat,  $p_1^* \mathcal{F} \subseteq p_1^* \mathcal{F}'$ . Hence we have to show that the map  $\phi = \theta|_{\mathcal{F}} : \mathcal{F} \rightarrow p_1^* \mathcal{F}'$  factors through  $p_1^* \mathcal{F}$ . Since the assertion is local, we may check it on an open affine formal subscheme  $\mathcal{U} = \text{Spf } A \subseteq \mathcal{X}$ . Let  $F_A, F'_A, E_A, B(i)$  be as in the previous paragraph, and denote the composite  $B(1) \xrightarrow{p_{13}^*} B(2) \cong B(1) \hat{\otimes} B(1)$  by  $\delta$ . Then, by cocycle condition, the following diagram is commutative:

$$(1.1) \quad \begin{array}{ccc} E_A & \xrightarrow{\theta} & E_A \otimes_A B(1) \\ \theta \downarrow & & \downarrow \text{id} \otimes \delta \\ E_A \otimes_A B(1) & \xrightarrow{\theta \otimes \text{id}} & E_A \otimes_A (B(1) \hat{\otimes} B(1)). \end{array}$$

Consider the following diagram:

$$\begin{array}{ccccc}
F_A \otimes_A B(1) & \xrightarrow{\phi \otimes \text{id}} & F'_A \otimes_A B(1) \otimes_A B(1) & & \\
\downarrow & & \downarrow & & \\
F_A & \xrightarrow{\phi} & F'_A \otimes_A B(1) & \xrightarrow{\phi' \otimes \text{id}} & E_A \otimes_A B(1) \otimes_A B(1) \\
\downarrow & & \downarrow & & \parallel \\
E_A & \xrightarrow{\theta} & E_A \otimes_A B(1) & \xrightarrow{\theta \otimes \text{id}} & E_A \otimes_A B(1) \otimes_A B(1).
\end{array}$$

By definition, the square on the bottom left is Cartesian. Since  $p_1$  is flat, the large rectangle on the right is also Cartesian. Thus it suffices to prove that the composition

$$F_A \rightarrow E_A \xrightarrow{\theta} E_A \otimes_A B(1) \xrightarrow{\theta \otimes \text{id}} E_A \otimes_A B(1) \otimes_A B(1) \xrightarrow{\text{pr} \otimes \text{id}} E_A/F'_A \otimes_A B(1) \otimes_A B(1)$$

is the zero map. Since  $E_A/F'_A$  is  $p$ -torsion, the natural map

$$E_A/F'_A \otimes_A B(1) \otimes_A B(1) \longrightarrow E_A/F'_A \otimes_A (B(1) \hat{\otimes}_A B(1))$$

is isomorphic, so it suffices to show that our map becomes zero after we follow it with this isomorphism. If  $x \in F_A$ , then  $\theta(x) \in F'_A \otimes_A B(1)$  holds, and so  $(\theta \otimes \text{id})(\theta(x)) = (\text{id} \otimes \delta)(\theta(x)) \in F'_A \otimes_A (B(1) \hat{\otimes}_A B(1))$  by the commutative diagram (1.1). This proves the assertion that  $\theta(\mathcal{F})$  is contained in  $p_1^* \mathcal{F}$ .

By the above assertion,  $\theta$  induces a morphism  $\epsilon_{\mathcal{F}} : p_2^* \mathcal{F} \longrightarrow p_1^* \mathcal{F}$  with  $\mathbb{Q} \otimes \epsilon_{\mathcal{F}} = \epsilon$ . Finally we prove that  $\epsilon_{\mathcal{F}}$  is an isomorphism. To see this, we may work on an open affine formal subscheme  $\mathcal{U} = \text{Spf } A \subseteq \mathcal{X}$ . Let  $F_A, B(1)$  be as before and let us consider the morphism  $\epsilon_A := \epsilon_{\mathcal{F}}|_{\mathcal{U}} : B(1) \otimes_A F_A \xrightarrow{\cong} F_A \otimes_A B(1)$ . Let us put  $C := \text{Coker}(\epsilon_A)$ . It suffices to prove that  $C/p^n C = 0$  for any  $n$ . Note that  $A \otimes_{\Delta^*, B(1)} \epsilon_A$  is, by definition, the identity map  $F_A \longrightarrow F_A$ . So we have  $A \otimes_{\Delta^*, B(1)} C = 0$ , hence  $(A/p^n A) \otimes_{\Delta^*, B(1)/p^n B(1)} (C/p^n C) = 0$ . Since  $\text{Ker}(\Delta^* : B(1)/p^n B(1) \longrightarrow A/p^n A)$  is a nil-ideal and  $C/p^n C$  is finitely generated, it implies that  $C/p^n C = 0$ . So  $\epsilon_A$  is an isomorphism and we are done.  $\square$

Let us go back to the situation before Proposition 1.1. We can prove a slightly stronger assertion than Proposition 1.1.

**Proposition 1.4.** *Let the notations be as above. Then, we can take  $\mathcal{E}^{(m)}$  in Proposition 1.1 so that  $E^{(m)} = \mathcal{E}^{(m)}/\varpi \mathcal{E}^{(m)}$  is a torsion free  $\mathcal{O}_X$ -module.*

*Proof.* Let  $\mathfrak{E}^{(m)}$  be the coherent  $\mathcal{O}_{\mathfrak{X}}$ -module corresponding to  $\mathcal{E}^{(m)}$  via GFGA. Also, let  $\mathcal{D}_{\mathfrak{X},n}^{(m)}$  be the coherent  $\mathcal{O}_{\mathfrak{X}}$ -module corresponding to  $\mathcal{D}_{\mathcal{X},n}^{(m)}$  via GFGA and put  $\mathcal{D}_{\mathfrak{X}}^{(m)} := \bigcup_n \mathcal{D}_{\mathfrak{X},n}^{(m)}$ . (Then it forms a ring.) Then  $\mathfrak{E}^{(m)}$  has a structure of  $\mathcal{D}_{\mathfrak{X}}^{(m)}$ -module.

Since  $\mathbf{E}$  is locally free and  $\mathcal{E}^{(m)}$  is  $p$ -torsion free,  $\mathfrak{E}^{(m)}$  is locally free on an open subscheme  $\mathfrak{U}$  of  $\mathfrak{X}$  with  $\text{codim}(\mathfrak{X} \setminus \mathfrak{U}, \mathfrak{X}) \geq 2$ . Let  $j : \mathfrak{U} \hookrightarrow \mathfrak{X}$  be the inclusion. Then,

if we replace  $\mathfrak{E}^{(m)}$  by  $j_* j^* \mathfrak{E}^{(m)}$ , it still has a structure of  $\mathcal{D}_{\mathfrak{X}}^{(m)}$ -module and it is a reflexive  $\mathcal{O}_{\mathfrak{X}}$ -module. So it satisfies Serre's condition  $S_2$ . Then  $E^{(m)} = \mathfrak{E}^{(m)} / \varpi \mathfrak{E}^{(m)}$  satisfies Serre's condition  $S_1$  and so it is torsion free. Then, the  $p$ -adic completion  $\mathcal{E}^{(m)}$  of  $\mathfrak{E}^{(m)}$  satisfies the condition of the proposition, because the  $p$ -adic completion of  $\mathcal{D}_{\mathfrak{X}}^{(m)}$  is equal to  $\widehat{\mathcal{D}}_{\mathcal{X}}^{(m)}$ .  $\square$

Let the situation be as before Proposition 1.1 and take a  $p$ -torsion free  $\mathcal{O}_{\mathcal{X}}$ -coherent quasi-nilpotent  $\widehat{\mathcal{D}}_{\mathcal{X}}^{(m)}$ -module  $\mathcal{E}^{(m)}$  with  $\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{E}^{(m)} = \mathcal{E}$  as in Proposition 1.4. Also, let  $\mathfrak{E}^{(m)}$ ,  $E^{(m)}$  be as in (the proof of) Proposition 1.4.

By Berthelot's theory Frobenius descent ([4, 2.3.7]), there exist equivalences (called the level raising Frobenius pullback functor)

$$(1.2) \quad \mathbb{F}^{i*} : \left( \begin{array}{c} \mathcal{O}_{\mathcal{X}^{[i]}}\text{-coherent} \\ \text{quasi-nilpotent } \widehat{\mathcal{D}}_{\mathcal{X}^{[i]}}^{(m-i)}\text{-modules} \end{array} \right) \xrightarrow{\cong} \left( \begin{array}{c} \mathcal{O}_{\mathcal{X}}\text{-coherent} \\ \text{quasi-nilpotent } \widehat{\mathcal{D}}_{\mathcal{X}}^{(m)}\text{-modules} \end{array} \right)$$

for  $0 \leq i \leq m - \mathfrak{e}$  induced by  $i$ -times iteration  $F^i$  of relative Frobenius. For precise definition of  $\mathbb{F}^{i*}$ , see [4, 2.2.6(ii)] or Section 2 in this paper. (We wrote the level raising Frobenius pullback functor by  $\mathbb{F}^{i*}$  to distinguish it from the pullback  $F^{i*}$  of  $\mathcal{O}_{\mathcal{X}}$ -modules by relative Frobenius. However, for  $\mathcal{O}_{\mathcal{X}^{[i]}}$ -coherent quasi-nilpotent  $\widehat{\mathcal{D}}_{\mathcal{X}^{[i]}}^{(m-i)}$ -modules  $\mathcal{F}$  with  $\varpi \mathcal{F} = 0$ , they are the same if we forget the  $\widehat{\mathcal{D}}_{\mathcal{X}}^{(m)}$ -module structure on  $\mathbb{F}^{i*} \mathcal{F}$ .) For an  $\mathcal{O}_{\mathcal{X}}$ -coherent quasi-nilpotent  $\widehat{\mathcal{D}}_{\mathcal{X}}^{(m)}$ -module  $\mathcal{F}$ , we call the  $\mathcal{O}_{\mathcal{X}^{[i]}}$ -coherent quasi-nilpotent  $\widehat{\mathcal{D}}_{\mathcal{X}^{[i]}}^{(m-i)}$ -module  $\mathcal{F}'$  satisfying  $\mathbb{F}^{i*} \mathcal{F}' = \mathcal{F}$  the  $i$ -th Frobenius antecedent of  $\mathcal{E}^{(m)}$ .

For  $0 \leq i \leq m - \mathfrak{e}$ , let  $\mathcal{E}^{(m)[i]}$ ,  $E^{(m)[i]}$  be the  $i$ -th Frobenius antecedent of  $\mathcal{E}^{(m)}$ ,  $E^{(m)}$ , respectively. (Then  $E^{(m)[i]} = \mathcal{E}^{(m)[i]} / \varpi \mathcal{E}^{(m)[i]}$ .) Let us denote the coherent  $\mathcal{O}_{\mathfrak{X}^{[i]}}$ -module corresponding to  $\mathcal{E}^{(m)[i]}$  via GFGA by  $\mathfrak{E}^{(m)[i]}$ . Then we have the following:

**Proposition 1.5.** *The sheaf  $\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{E}^{(m)[i]}$  does not depend on the choice of  $\mathcal{E}^{(m)}$  as in Proposition 1.1, and does not depend on  $m \geq i + \mathfrak{e}$  either. (Hence, by GFGA, the same is true for the restriction of  $\mathfrak{E}^{(m)[i]}$  to  $\mathbf{X}^{[i]}$ .)*

*Proof.* If we choose another  $\mathcal{E}^{(m)}$  which we denote by  $\mathcal{F}^{(m)}$ , we have morphisms  $\alpha : \mathcal{E}^{(m)} \rightarrow \mathcal{F}^{(m)}$ ,  $\beta : \mathcal{F}^{(m)} \rightarrow \mathcal{E}^{(m)}$  with  $\beta \circ \alpha = p^N$ ,  $\alpha \circ \beta = p^N$  for some  $N \in \mathbb{N}$ . Let us denote the  $i$ -th Frobenius antecedent of  $\mathcal{F}^{(m)}$  by  $\mathcal{F}^{(m)[i]}$ . Then, since  $\mathbb{F}^{i*}$  is an equivalence, the maps  $\alpha, \beta$  induce the maps  $\alpha' : \mathcal{E}^{(m)[i]} \rightarrow \mathcal{F}^{(m)[i]}$ ,  $\beta' : \mathcal{F}^{(m)[i]} \rightarrow \mathcal{E}^{(m)[i]}$  with  $\beta' \circ \alpha' = p^N$ ,  $\alpha' \circ \beta' = p^N$ . So  $\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{E}^{(m)[i]} = \mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{F}^{(m)[i]}$ .

For  $i + \mathfrak{e} \leq m \leq m'$ ,  $\mathcal{E}^{(m')}$  as in Proposition 1.1 can be regarded also as a  $\widehat{\mathcal{D}}_{\mathcal{X}}^{(m)}$ -module via the restriction by the canonical map  $\widehat{\mathcal{D}}_{\mathcal{X}}^{(m)} \rightarrow \widehat{\mathcal{D}}_{\mathcal{X}}^{(m')}$ , and the functor  $\mathbb{F}^{i*}$  is compatible with it. So we see that  $\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{E}^{(m)[i]}$  does not depend on  $m$ .  $\square$

So, in the sequel, we denote the sheaf  $\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{E}^{(m)[i]}$  by  $\mathcal{E}^{[i]}$  and the restriction of  $\mathfrak{E}^{(m)[i]}$  to  $\mathbf{X}^{[i]}$  by  $\mathbf{E}^{[i]}$ . Note that these are defined for all  $i \in \mathbb{N}$ . Note also that

$\mathcal{E}^{[i]} = \mathbb{Q} \otimes \mathcal{E}^{(m)[i]}$  has a structure of  $\widehat{\mathcal{D}}_{\mathcal{X}^{[i]}, \mathbb{Q}}^{(m)}$ -modules ( $m \geq i + \mathfrak{e}$ ) which are compatible with respect to  $m$ , which induces a structure of  $\mathcal{D}_{\mathcal{X}^{[i]}, \mathbb{Q}}^\dagger$ -module.

**Proposition 1.6.** *For any  $i \in \mathbb{N}$ , the reduced Hilbert polynomial  $p_{\mathbf{E}^{[i]}}$  of  $\mathbf{E}^{[i]}$  is equal to  $p_{\mathcal{O}_X}$ . In particular,  $\mu(\mathbf{E}^{[i]}) = 0$ .*

*Proof.* We give two proofs. First, if we define  $\mathcal{D}_{\mathfrak{X}^{[i]}}^{(m)}$  as in the proof of Proposition 1.4 (with  $\mathfrak{X}$  replaced by  $\mathfrak{X}^{[i]}$ ),  $\mathfrak{E}^{(m)[i]}$  has a structure of  $\mathcal{D}_{\mathfrak{X}^{[i]}}^{(m)}$ -module. Also, one can see that the restriction of  $\mathcal{D}_{\mathfrak{X}^{[i]}}^{(m)}$  to  $\mathbf{X}^{[i]}$  is nothing but the usual  $D$ -module  $\mathcal{D}_{\mathbf{X}^{[i]}}$  on  $\mathbf{X}^{[i]}$ . So  $\mathbf{E}^{[i]}$  has a structure of  $\mathcal{D}_{\mathbf{X}^{[i]}}$ -module. Then it is well-known (e.g., [10, 2.3.1]) that the Chern classes of non-zero degree of  $\mathbf{E}^{[i]}$  vanish, and so  $p_{\mathbf{E}^{[i]}}$  is equal to  $p_{\mathcal{O}_{\mathbf{X}^{[i]}}} = p_{\mathcal{O}_X} = p_{\mathcal{O}_X}$  by Riemann-Roch.

The second proof is analogous to the proof in [5, Lemma 2.1]. By Riemann-Roch, it suffices to prove the following claim: For any  $0 \leq j \leq d - 1$ , for any class  $\xi \in CH_j(\mathbf{X}^{[i]})$  and for any homogeneous polynomial  $\gamma_{d-j}$  of degree  $d - j$  with rational coefficients in the Chern class, we have  $\xi_{\text{et}} \cdot \gamma_{d-j}(\mathbf{E}^{[i]})_{\text{et}} = 0$ , where  $\xi_{\text{et}}, \gamma_{d-j}(\mathbf{E}^{[i]})_{\text{et}}$  are the class of  $\xi, \gamma_{d-j}(\mathbf{E}^{[i]})$  considered in  $l$ -adic etale cohomology  $H_{\text{et}}^*(\overline{\mathbf{X}}^{[i]}, \mathbb{Q}_l) \cong H_{\text{et}}^*(X^{[i]}, \mathbb{Q}_l)$ . (Here  $\overline{\mathbf{X}}^{[i]}$  is the geometric fiber of  $\mathbf{X}^{[i]}$ .) If we take any  $i \leq i' \leq m - \mathfrak{e}$ , we have

$$\begin{aligned} \xi_{\text{et}} \cdot \gamma_{d-j}(\mathbf{E}^{[i]})_{\text{et}} &= \xi_{\text{et}} \cdot \gamma_{d-j}(E^{(m)[i]})_{\text{et}} = \xi_{\text{et}} \cdot \gamma_{d-j}(F^{i'-i*} E^{(m)[i']})_{\text{et}} \\ &= p^{i'-i} \xi_{\text{et}} \cdot \gamma_{d-j}(E^{(m)[i']})_{\text{et}} = p^{i'-i} \xi_{\text{et}} \cdot \gamma_{d-j}(\mathbf{E}^{[i']})_{\text{et}}. \end{aligned}$$

Since the above equality holds for any  $i \leq i'$  and  $\xi_{\text{et}} \cdot \gamma_{d-j}(\mathbf{E}^{[i']})_{\text{et}}$  ( $i \leq i'$ ) are rational numbers with bounded denominator (depending only on  $\xi$  and  $\gamma_{d-j}$ ), we see the equality  $\xi_{\text{et}} \cdot \gamma_{d-j}(\mathbf{E}^{[i]})_{\text{et}} = 0$ .  $\square$

We say that a convergent isocrystal  $\mathcal{E}$  on  $X/K$  is trivial if it is isomorphic to a finite direct sum of the structure convergent isocrystal  $\mathcal{O}_{X/K}$  (which corresponds to the  $\mathcal{D}_{\mathcal{X}, \mathbb{Q}}^\dagger$ -module  $\mathcal{O}_{\mathcal{X}, \mathbb{Q}}$  defined by the canonical action via derivation).

To proceed further, we have to impose certain conditions of (semi)stability. We state our first result, which assumes a certain stability condition:

**Theorem 1.7.** *Let  $X$  be as above and assume that its etale fundamental group  $\pi_1^{\text{et}}(X)$  is trivial. Then, any convergent isocrystal  $\mathcal{E}$  on  $X/K$  which satisfies the following condition (A) is trivial:*

(A): *For infinitely many natural numbers  $m$ , there exists some  $i = i(m) \in \mathbb{N}$  and some  $p$ -torsion free  $\mathcal{O}_X$ -coherent quasi-nilpotent  $\widehat{\mathcal{D}}_{\mathcal{X}^{[i]}}^{(m)}$ -module  $\mathcal{G}^{[i](m)}$  with  $\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{G}^{[i](m)} = \mathcal{E}^{[i]}$  as  $\widehat{\mathcal{D}}_{\mathcal{X}^{[i]}, \mathbb{Q}}^{(m)}$ -modules such that  $G^{[i](m)} := \mathcal{G}^{[i](m)} / \varpi \mathcal{G}^{[i](m)}$  is  $\mu$ -stable as  $\mathcal{O}_{X^{[i]}}$ -module.*

**Corollary 1.8.** *Let  $X$  be as above and assume that its etale fundamental group  $\pi_1^{\text{et}}(X)$  is trivial. Then any convergent isocrystal  $\mathcal{E}$  on  $X/K$  of rank 1 is trivial.*



*Proof.* If the rank of  $\mathcal{E}$  is equal to 1,  $\mathcal{E}^{(m)}$  ( $m \in \mathbb{N}$ ) in Proposition 1.4 satisfies the condition (A) (for  $\mathcal{G}^{[i](m)}$  with  $i = 0$ ) because any torsion free  $\mathcal{O}_X$ -module of generic rank 1 is  $\mu$ -stable.  $\square$

We have a similar result under certain assumption of semistability:

**Theorem 1.9.** *Let  $X$  be as above and assume that its etale fundamental group  $\pi_1^{\text{et}}(X)$  is trivial. Then any convergent isocrystal  $\mathcal{E}$  on  $X/K$  which satisfies the following condition (B) is trivial:*

(B): *For infinitely many natural numbers  $m$ , there exists some  $i = i(m), l = l(m) \in \mathbb{N}$  with  $l(m) \leq m - \mathfrak{e}$ ,  $l(m) \rightarrow \infty$  ( $m \rightarrow \infty$ ) and some  $p$ -torsion free  $\mathcal{O}_{\mathcal{X}^{[i]}}$ -coherent quasi-nilpotent  $\widehat{\mathcal{D}}_{\mathcal{X}^{[i]}}^{(m)}$ -module  $\mathcal{G}^{[i](m)}$  with  $\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{G}^{[i](m)} = \mathcal{E}^{[i]}$  as  $\widehat{\mathcal{D}}_{\mathcal{X}^{[i]}, \mathbb{Q}}^{(m)}$ -modules such that, for any  $0 \leq j \leq l$ , the  $j$ -th Frobenius antecedent  $G^{[i](m)[j]}$  of  $G^{[i](m)} := \mathcal{G}^{[i](m)} / \varpi \mathcal{G}^{[i](m)}$  is semistable as  $\mathcal{O}_X$ -module.*

We have the  $\sigma_K^{n*}$ -linear pullback functor

$$F_{\text{abs}}^{n*} : \left( \begin{array}{c} \text{convergent} \\ \text{isocrystals on } X/K \end{array} \right) \xrightarrow{\cong} \left( \begin{array}{c} \text{convergent} \\ \text{isocrystals on } X/K \end{array} \right)$$

induced by  $F_{\text{abs}}^n$ . In terms of arithmetic  $D$ -modules, it is written as the composite of the  $\sigma_K^{n*}$ -linear pullback functor

$$\pi^{n*} : \left( \begin{array}{c} \mathcal{O}_{\mathcal{X}, \mathbb{Q}}\text{-coherent} \\ \mathcal{D}_{\mathcal{X}}^{\dagger}\text{-modules} \end{array} \right) \xrightarrow{\cong} \left( \begin{array}{c} \mathcal{O}_{\mathcal{X}^{[n]}, \mathbb{Q}}\text{-coherent} \\ \mathcal{D}_{\mathcal{X}^{[n]}}^{\dagger}\text{-modules} \end{array} \right)$$

induced by  $\pi^n$  and the functor

$$\mathbb{F}_{\mathbb{Q}}^{n*} : \left( \begin{array}{c} \mathcal{O}_{\mathcal{X}^{[n]}, \mathbb{Q}}\text{-coherent} \\ \mathcal{D}_{\mathcal{X}^{[n]}}^{\dagger}\text{-modules} \end{array} \right) \xrightarrow{\cong} \left( \begin{array}{c} \mathcal{O}_{\mathcal{X}, \mathbb{Q}}\text{-coherent} \\ \mathcal{D}_{\mathcal{X}}^{\dagger}\text{-modules} \end{array} \right)$$

induced by the  $\mathbb{Q}$ -linearization of the level raising Frobenius pullback functors

$$\mathbb{F}^{n*} : \left( \begin{array}{c} \mathcal{O}_{\mathcal{X}^{[n]}}\text{-coherent} \\ \text{quasi-nilpotent } \widehat{\mathcal{D}}_{\mathcal{X}^{[n]}}^{(m)}\text{-modules} \end{array} \right) \xrightarrow{\cong} \left( \begin{array}{c} \mathcal{O}_{\mathcal{X}}\text{-coherent} \\ \text{quasi-nilpotent } \widehat{\mathcal{D}}_{\mathcal{X}}^{(m+n)}\text{-modules} \end{array} \right)$$

for  $m \gg 0$ . Note that the functors  $\pi^{n*}, \mathbb{F}_{\mathbb{Q}}^{n*}$  can be written as a certain iteration of the functors  $\pi^*, \mathbb{F}_{\mathbb{Q}}^*$  in the case  $n = 1$ , and they are ‘commutative’ in suitable sense as long as they are defined.

In this paper, a convergent  $F$ -isocrystal on  $X/K$  is a pair  $(\mathcal{E}, \Phi)$  consisting of a convergent isocrystal  $\mathcal{E}$  on  $X/K$  endowed with an isomorphism  $\Phi : F_{\text{abs}}^{n*} \mathcal{E} \xrightarrow{\cong} \mathcal{E}$  for some  $n \in \mathbb{N}$ .

Then we have the following corollary of Theorem 1.9:

**Corollary 1.10.** *Let  $X$  be as above and assume that its etale fundamental group  $\pi_1^{\text{et}}(X)$  is trivial. Then, for any convergent  $F$ -isocrystal  $(\mathcal{E}, \Phi)$  on  $X/K$  which satisfies the following condition (C),  $\mathcal{E}$  is trivial as a convergent isocrystal:*

(C): *There exists some  $i \in \mathbb{N}$  and some  $p$ -torsion free  $\mathcal{O}_{\mathcal{X}^{[i]}}$ -coherent quasi-nilpotent  $\widehat{\mathcal{D}}_{\mathcal{X}^{[i]}}^{(\mathfrak{e})}$ -module  $\mathcal{G}^{[i](\mathfrak{e})}$  with  $\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{G}^{[i](\mathfrak{e})} = \mathcal{E}^{[i]}$  as  $\widehat{\mathcal{D}}_{\mathcal{X}^{[i], \mathbb{Q}}}^{(\mathfrak{e})}$ -modules such that  $G^{[i](\mathfrak{e})} := \mathcal{G}^{[i](\mathfrak{e})} / \varpi \mathcal{G}^{[i](\mathfrak{e})}$  is strongly semistable as  $\mathcal{O}_X$ -module.*

*Proof.* Because we have  $\mathbb{F}_{\mathbb{Q}}^i * \mathcal{E}^{[i]} = \mathcal{E}$ ,  $\mathbb{F}_{\mathbb{Q}}^i * F_{\text{abs}}^n * \mathcal{E}^{[i]} = \mathbb{F}_{\mathbb{Q}}^i * \mathbb{F}_{\mathbb{Q}}^n * \pi^{n*} \mathcal{E}^{[i]} = \mathbb{F}_{\mathbb{Q}}^n * \pi^{n*} \mathbb{F}_{\mathbb{Q}}^i * \mathcal{E}^{[i]} = F_{\text{abs}}^n * \mathcal{E}$ ,  $\mathcal{E}^{[i]}$  admits a structure of convergent  $F$ -isocrystal by some isomorphism  $\Psi : F_{\text{abs}}^n * \mathcal{E}^{[i]} \xrightarrow{\cong} \mathcal{E}^{[i]}$ . Then, for  $m \in n\mathbb{N} + \mathfrak{e}$ , we put  $\mathcal{G}^{[i](m)} := \mathbb{F}^{m-\mathfrak{e}*} \pi^{m-\mathfrak{e}*} \mathcal{G}^{[i](\mathfrak{e})}$ ,  $G^{[i](m)} := \mathcal{G}^{[i](m)} / \varpi \mathcal{G}^{[i](m)}$ . Then we have  $\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{G}^{[i](m)} = F_{\text{abs}}^{m-\mathfrak{e}*} \mathcal{E}^{[i]} \cong \mathcal{E}^{[i]}$  as  $\widehat{\mathcal{D}}_{\mathcal{X}^{[i], \mathbb{Q}}}^{(m)}$ -modules. Also, for  $0 \leq j \leq m - \mathfrak{e}$ , the  $j$ -th Frobenius antecedent of  $G^{[i](m)}$  is equal to  $F^{m-\mathfrak{e}-j*} \pi^{m-\mathfrak{e}*} G^{[i](\mathfrak{e})}$ , which is semistable by the strong semistability of  $G^{[i](\mathfrak{e})}$ . So  $\mathcal{E}$  satisfies the condition (B) in Theorem 1.9 and so it is trivial as a convergent isocrystal.  $\square$

We restate the corollary in the case  $V = W(k), i = 0$  for the convenience to the reader:

**Corollary 1.11.** *Let  $X$  be a projective smooth variety over  $k$  which is liftable to a projective smooth formal scheme  $\mathcal{X}$  over  $\text{Spf } W(k)$ . Also, assume that the etale fundamental group  $\pi_1^{\text{et}}(X)$  is trivial. Then, for any convergent  $F$ -isocrystal  $(\mathcal{E}, \Phi)$  on  $X/\text{Frac } W(k)$  which satisfies the following condition (D),  $\mathcal{E}$  is trivial as a convergent isocrystal:*

(D): *There exists some  $p$ -torsion free  $\mathcal{O}_{\mathcal{X}}$ -coherent quasi-nilpotent  $\widehat{\mathcal{D}}_{\mathcal{X}}^{(0)}$ -module  $\mathcal{G}$  with  $\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{G} = \mathcal{E}$  as  $\widehat{\mathcal{D}}_{\mathcal{X}, \mathbb{Q}}^{(0)}$ -modules such that  $G := \mathcal{G}/p\mathcal{G}$  is strongly semistable as  $\mathcal{O}_X$ -module.*

## 2 Proofs

In this section, we give proofs of Theorems 1.7, 1.9. So, in this section, let  $X$  be as in the previous section and assume moreover that  $\pi_1^{\text{et}}(X)$  is trivial.

**Proposition 2.1.** *To prove theorems, we can enlarge  $k$  so that  $k$  is uncountable.*

*Proof.* Let  $k'$  be an uncountable algebraically closed field containing  $k$  and let  $K'$  be the fraction field of  $V \otimes_{W(k)} W(k')$ . Put  $X' := X \otimes_k k'$  and denote the pull-back of  $\mathcal{E}$  to  $X'$  by  $\mathcal{E}'$ , which is a convergent isocrystal on  $X'/K'$ . Also, put  $\mathbf{X}' := \mathbf{X} \otimes_K K'$ , let  $\mathbf{E}'$  be the pullback of  $\mathbf{E}$  to  $\mathbf{X}'$  and let  $\mathcal{D}_{\mathbf{X}}, \mathcal{D}_{\mathbf{X}'}$  be usual  $D$ -modules of  $\mathbf{X}, \mathbf{X}'$ . Then, as we have seen in the proof of Proposition 1.6,  $\mathbf{E}$  admits naturally a structure of  $\mathcal{D}_{\mathbf{X}}$ -module and  $\mathbf{E}'$  admits a structure of  $\mathcal{D}_{\mathbf{X}'}$ -module, which is the pullback of the  $\mathcal{D}_{\mathbf{X}}$ -module structure of  $\mathbf{E}$ .

If we assume that the theorem is true for  $X'$  and  $\mathcal{E}'$ , we have  $\dim_{K'} H_{\text{rig}}^0(X'/K', \mathcal{E}') = r$ . Because

$$H_{\text{rig}}^0(X'/K', \mathcal{E}') = H_{\text{dR}}^0(\mathbf{X}'/K', \mathbf{E}') = H_{\text{dR}}^0(\mathbf{X}/K, \mathbf{E}) \otimes_K K' = H_{\text{rig}}^0(X/K, \mathcal{E}) \otimes_K K'$$

by the comparison theorem of Ogus ([14], [15]), we have the equality  $\dim_K H_{\text{rig}}^0(X/K, \mathcal{E}) = r$ . So  $\mathcal{E}$  is also trivial.  $\square$

So, in the sequel, we assume that  $k$  is uncountable.

The following proposition, which uses Gieseker conjecture (proven by Esnault-Mehta [5]) in  $\mu$ -stable case, is the first step for the proof:

**Proposition 2.2.** *Let  $r$  be a positive integer. Then there exists a positive integer  $a = a(r)$  satisfying the following condition: For any sequence of stable sheaves  $\{E_i\}_{i=0}^a$  of length  $a$  with  $E_a$  on  $X^{[a+j]}$  for some  $j \geq 0$ ,  $\text{rank } E_0 \leq r$ ,  $p_{E_i} = p_{\mathcal{O}_X}$  ( $0 \leq i \leq a$ ) and  $F^*E_{i+1} = E_i$  ( $0 \leq i \leq a-1$ ),  $E_0$  is isomorphic to  $\mathcal{O}_{X^{[j]}}$ .*

*Proof.* For  $1 \leq s \leq r$  and  $n \geq 0$ , let  $M_s^{[n+j]}$  be the moduli of stable sheaves on  $X^{[n+j]}$  with rank  $s$  and reduced Hilbert polynomial  $p_{\mathcal{O}_X}$ , which is constructed by Adrian Langer ([1], [2]). It is a quasi-projective scheme over  $k$ . Also, let  $M_{s,\circ}^{[n+j]}$  be the open subscheme consisting of stable sheaves  $G$  such that  $F^{n*}G$  remains stable. (This is known to be an open condition. See discussion in the beginning of [5, §3].) The pull-back by  $F$  induces the morphisms called Verschiebungs

$$\dots \longrightarrow M_{s,\circ}^{[2+j]} \xrightarrow{\mathbb{V}} M_{s,\circ}^{[1+j]} \xrightarrow{\mathbb{V}} M_s^{[j]}.$$

Let  $\text{Im } \mathbb{V}^n$  be the image of  $\mathbb{V}^n : M_{s,\circ}^{[n+j]} \longrightarrow M_s^{[j]}$ , which is a constructible set of  $M_s^{[j]}$ . Then,  $\dim \text{Im } \mathbb{V}^n$  is stable for  $n \gg 0$ , which we denote by  $f$ . Assume  $f > 0$ . Then the generic point of some irreducible closed subscheme of dimension  $f$  remains contained in  $\text{Im } \mathbb{V}^n$  ( $n \in \mathbb{N}$ ). Pick such an irreducible closed subscheme and denote it by  $C$ . Then  $C \cap \text{Im } \mathbb{V}^n$  is non-empty for any  $n \in \mathbb{N}$  and it contains an open subscheme of  $C$ . So there exists a closed subscheme  $D_n \subsetneq C$  of smaller dimension such that  $C \cap \text{Im } \mathbb{V}^n \supseteq C \setminus D_n$ . Then  $C \cap (\bigcap_n \text{Im } \mathbb{V}^n) \supseteq C \setminus (\bigcup_n D_n)$ . So it contains at least two  $k$ -rational points  $P, P'$ , because  $k$  is uncountable. Since  $P, P'$  are  $k$ -rational points of  $\bigcap_n \text{Im } \mathbb{V}^n$ , they induce two non-isomorphic stratified sheaves on  $X^{[j]}$ . This contradicts (the  $\mu$ -stable case of) the Gieseker conjecture proven by Esnault-Mehta. So  $\text{Im } \mathbb{V}^n$  consists of finite set of points (possibly empty) for some  $n$ . Then, since  $\bigcap_n \text{Im } \mathbb{V}^n$  is empty (if  $s \geq 2$ ) or one point corresponding to  $\mathcal{O}_{X^{[j]}}$  (if  $s = 1$ ) by (the  $\mu$ -stable case of) Gieseker conjecture, it is equal to  $\text{Im } \mathbb{V}^{a(s)}$  for some  $a(s) \in \mathbb{N}$ . Let us define  $a$  to be the maximum of  $a(s)$  ( $s \leq r$ ). Then, if we are given a sequence  $\{E_i\}_{i=0}^a$  as in the statement of the proposition with  $s := \text{rank } E_0 \leq r$ ,  $E_0$  defines a  $k$ -rational point of  $\text{Im } \mathbb{V}^{a(s)} \subseteq M_s^{[j]}$ . Then  $s$  should be equal to 1 and  $E_0$  should be isomorphic to  $\mathcal{O}_{X^{[j]}}$ .  $\square$

**Remark 2.3.** There is a mistake in [5, Proposition 2.3], but it is fixed in [6]. We also point out that this mistake occurs in the discussion of reducing Gieseker's conjecture to that in  $\mu$ -stable case. Because we used Gieseker's conjecture only in  $\mu$ -stable case, we do not need the correction given in [6].

For the proof of Theorem 1.9, we need the following proposition, which proves the triviality of locally free sheaves of higher rank in certain situation.

**Proposition 2.4.** *There exists a positive integer  $b$  satisfying the following condition: For any sequence of locally free sheaves  $\{E_i\}_{i=0}^{b(r-1)}$  of length  $b(r-1)$  on  $X$  with  $E_{b(r-1)}$  on  $X^{[b(r-1)+j]}$  for some  $j \geq 0$ ,  $\text{rank } E_0 = r$ ,  $F^*E_{i+1} = E_i$  ( $0 \leq i \leq b(r-1)-1$ ) such that  $E_{b(r-1)}$  is an iterated extension of  $\mathcal{O}_{X^{[b(r-1)+j]}}$ ,  $E_0$  is isomorphic to  $\mathcal{O}_{X^{[j]}}$ .*

*Proof.* The proof is similar to that in [5, Proposition 2.4]. By [13, Corollary in p.143], we have the decomposition  $H^1(X^{[n]}, \mathcal{O}_{X^{[n]}}) = H^1(X^{[n]}, \mathcal{O}_{X^{[n]}})_{\text{nilp}} \oplus H^1(X^{[n]}, \mathcal{O}_{X^{[n]}})_{\text{ss}}$  of  $H^1(X^{[n]}, \mathcal{O}_{X^{[n]}})$  into the part  $H^1(X^{[n]}, \mathcal{O}_{X^{[n]}})_{\text{nilp}}$  where the absolute Frobenius  $F_{\text{abs}}^*$  acts nilpotently and the part  $H^1(X^{[n]}, \mathcal{O}_{X^{[n]}})_{\text{ss}}$  where the absolute Frobenius  $F_{\text{abs}}^*$  acts bijectively. Also, we have

$$H^1(X^{[n]}, \mathcal{O}_{X^{[n]}})_{\text{ss}} = H_{\text{et}}^1(X^{[n]}, \mathbb{Z}/p\mathbb{Z}) = \text{Hom}(\pi_1^{\text{et}}(X^{[n]}), \mathbb{Z}/p\mathbb{Z}) = 0$$

and there exists some  $b \in \mathbb{N}$  such that  $(F_{\text{abs}}^*)^b$  acts by 0 on  $H^1(X^{[n]}, \mathcal{O}_{X^{[n]}})_{\text{nilp}}$ , since  $H^1(X^{[n]}, \mathcal{O}_{X^{[n]}})_{\text{nilp}}$  is finite-dimensional. So  $(F_{\text{abs}}^*)^b$  acts by 0 on  $H^1(X^{[n]}, \mathcal{O}_{X^{[n]}})$ . (Also, we can take  $b$  independently of  $n \in \mathbb{N}$  because  $X^{[n]}$  is isomorphic to  $X$  via  $\pi^n$ .) We prove the proposition for this choice of  $b$ .

By assumption on  $E_{b(r-1)}$ , there exists a filtration

$$0 = E_{b(r-1),0} \subset E_{b(r-1),1} \subset \cdots \subset E_{b(r-1),r} = E_{b(r-1)}$$

whose graded quotients are isomorphic to  $\mathcal{O}_{X^{[b(r-1)+j]}}$ . By pulling it back to  $E_i$  via  $F^{*b(r-1)-i}$ , we obtain the filtration

$$0 = E_{i,0} \subset E_{i,1} \subset \cdots \subset E_{i,r} = E_i$$

of  $E_i$  whose graded quotients are isomorphic to  $\mathcal{O}_{X^{[b(r-1)-i+j]}}$ . We prove that  $E_{b(r-l),l}$  is isomorphic to  $\mathcal{O}_{X^{[b(r-l)+j]}}^l$  by induction. Assume that  $E_{b(r-l),l} \cong \mathcal{O}_{X^{[b(r-l)+j]}}^l$ . Then, for  $b(r-l-1) \leq n \leq b(r-l)$ , consider the extension class  $e_n$  of the exact sequence

$$0 \longrightarrow E_{n,l} \longrightarrow E_{n,l+1} \longrightarrow \mathcal{O}_{X^{[n+j]}} \longrightarrow 0$$

in  $H^1(X^{[n+j]}, E_{n,l}) = H^1(X^{[n+j]}, \mathcal{O}_{X^{[n+j]}})^l$ . The family of classes  $\{e_n\}_n$  defines an element of the inverse limit of the diagram

$$H^1(X^{[b(r-l)+j]}, \mathcal{O}_{X^{[b(r-l)+j]}})^l \xrightarrow{F^*} \cdots \xrightarrow{F^*} H^1(X^{[b(r-l-1)+j]}, \mathcal{O}_{X^{[b(r-l-1)+j]}})^l$$

of length  $b$  whose last component is  $e_{b(r-l-1)}$ . By twisting by the absolute Frobenius on  $k$  (which is bijective), we obtain an element of the inverse limit of the diagram

$$H^1(X^{[b(r-l-1)+j]}, \mathcal{O}_{X^{[b(r-l-1)+j]}})^l \xrightarrow{F_{\text{abs}}^*} \cdots \xrightarrow{F_{\text{abs}}^*} H^1(X^{[b(r-l-1)+j]}, \mathcal{O}_{X^{[b(r-l-1)+j]}})^l$$

of length  $b$  whose last component is  $e_{b(r-l-1)}$ . Then, by definition of  $b$ ,  $e_{b(r-l-1)} = 0$ . So  $E_{b(r-l-1), l+1}$  is isomorphic to  $\mathcal{O}_X^{l+1}$  and we are done.  $\square$

So far, we treated the triviality of sheaves  $\mathcal{F}$  with  $\varpi\mathcal{F} = 0$ . To lift this triviality to the triviality of certain sheaves  $\mathcal{F}$  with  $\varpi^N\mathcal{F} = 0$  ( $N \geq 2$ ), we need to review the definition of the level raising Frobenius pullback functor

$$(2.1) \quad \mathbb{F}^{s*} : \left( \begin{array}{c} \mathcal{O}_{\mathcal{X}^{[s]}\text{-coherent}} \\ \text{quasi-nilpotent } \widehat{\mathcal{D}}_{\mathcal{X}^{[s]}}^{(m)}\text{-modules} \end{array} \right) \xrightarrow{\cong} \left( \begin{array}{c} \mathcal{O}_{\mathcal{X}\text{-coherent}} \\ \text{quasi-nilpotent } \widehat{\mathcal{D}}_{\mathcal{X}}^{(m+s)}\text{-modules} \end{array} \right)$$

for  $m \geq \mathfrak{e}$ .

**Remark 2.5.** Precisely speaking, we need the level raising Frobenius pullback functor of the form

$$(2.2) \quad \mathbb{F}^{s*} : \left( \begin{array}{c} \mathcal{O}_{\mathcal{X}^{[s+t]}\text{-coherent}} \\ \text{quasi-nilpotent } \widehat{\mathcal{D}}_{\mathcal{X}^{[s+t]}}^{(m)}\text{-modules} \end{array} \right) \xrightarrow{\cong} \left( \begin{array}{c} \mathcal{O}_{\mathcal{X}^{[t]}\text{-coherent}} \\ \text{quasi-nilpotent } \widehat{\mathcal{D}}_{\mathcal{X}^{[t]}}^{(m+s)}\text{-modules} \end{array} \right)$$

for  $m \geq \mathfrak{e}$ . However, to lighten the notation, we describe the definition in detail only in the case of the functor (2.1) (the case  $t = 0$ ).

First we give the definition in local situation. So let us forget the projectivity of  $X, \mathcal{X}$  for a while. Assume that there exists a local coordinate  $t_1, \dots, t_d$  of  $\mathcal{X}$  over  $\mathrm{Spf} V$ . (Then it induces a local coordinate of  $\mathcal{X}^{[s]}$  over  $\mathrm{Spf} V$ , which we also denote by  $t_1, \dots, t_d$ .) Also, take a lift  $\tilde{F}^s : \mathcal{X} \rightarrow \mathcal{X}^{[s]}$  of  $s$ -times iteration of the relative Frobenius morphism compatible with the morphism  $\mathrm{Spf} V \rightarrow \mathrm{Spf} V$  induced by  $\sigma_V^s$  which satisfies the equalities  $(\tilde{F}^s)^*(t_i) = t_i^{p^s}$  for  $1 \leq i \leq d$ . For  $m, n \in \mathbb{N}$  with  $m \geq \mathfrak{e}$ , let  $\mathcal{D}(1)_{(m)}, \mathcal{D}(1)_{(m)}^n$  be as in the previous section. Also, let  $\mathcal{D}(1)_{(m)}^{[s]}, \mathcal{D}(1)_{(m)}^{[s], n}$  be the pull-back of  $\mathcal{D}(1)_{(m)}, \mathcal{D}(1)_{(m)}^n$  by  $\pi^n$ , respectively. Then the homomorphism

$$(2.3) \quad (\tilde{F}^s \times \tilde{F}^s)^* : \mathcal{O}_{\mathcal{X}^{[s]} \times \mathcal{X}^{[s]}} \rightarrow \mathcal{O}_{\mathcal{X} \times \mathcal{X}}$$

naturally induces the homomorphisms

$$(2.4) \quad \mathcal{O}_{\mathcal{D}(1)_{(m)}^{[s], n}} \rightarrow \mathcal{O}_{\mathcal{D}(1)_{(m+s)}^n} \quad (n \in \mathbb{N}).$$

(Here we use the assumption  $m \geq \mathfrak{e}$ . See [4, 2.2.2].) If we take modulo  $p^N$ , take the dual, take the union with respect to  $n$  and take the inverse limit with respect to  $N \in \mathbb{N}$ , we obtain the homomorphism

$$(2.5) \quad \widehat{\mathcal{D}}_{\mathcal{X}}^{(m+s)} \rightarrow (\tilde{F}^s)^* \widehat{\mathcal{D}}_{\mathcal{X}^{[s]}}^{(m)}.$$

Then, for a  $\mathcal{O}_{\mathcal{X}^{[s]}\text{-coherent}}$  quasi-nilpotent  $\widehat{\mathcal{D}}_{\mathcal{X}^{[s]}}^{(m)}$ -module  $\mathcal{E}$ ,  $(\tilde{F}^s)^*\mathcal{E}$  admits an action of  $\widehat{\mathcal{D}}_{\mathcal{X}}^{(m+s)}$  via the map (2.5) and it becomes an  $\mathcal{O}_{\mathcal{X}\text{-coherent}}$  quasi-nilpotent  $\widehat{\mathcal{D}}_{\mathcal{X}}^{(m+s)}$ -module. This is the definition of the level raising Frobenius pullback functor (2.1)

in local situation. If we put  $\tau_i := 1 \otimes t_i - t_i \otimes 1$ , the homomorphism (2.3) is written as

$$(2.6) \quad \begin{aligned} \tau_i &\mapsto 1 \otimes t_i^{p^s} - t^{p^s} \otimes 1 \\ &= (\tau_i + t_i \otimes 1)^{p^s} - t_i^{p^s} \otimes 1 = \tau_i^{p^s} + \sum_{j=1}^{p^s-1} \binom{p^s}{j} t_i^{p^s-j} \tau_i^j. \end{aligned}$$

We calculate a part of  $\widehat{\mathcal{D}}_{\mathcal{X}}^{(m+s)}$ -action on  $(\widetilde{F}^s)^* \mathcal{E}$  when  $m = \mathfrak{e}$  and  $\mathcal{E}$  is torsion.

**Lemma 2.6.** *Let  $N \geq 1$ . Let the notations be as above with  $m = \mathfrak{e}$  (so  $\mathcal{E}$  is an  $\mathcal{O}_{\mathcal{X}}$ -coherent quasi-nilpotent  $\widehat{\mathcal{D}}_{\mathcal{X}^{[s]}}^{(\mathfrak{e})}$ -module) and assume that  $\varpi^N \mathcal{E} = 0$ . Also, let  $\{\partial_{\langle l \rangle}\}_{l \in \mathbb{N}^d}$  be the family of elements in  $\widehat{\mathcal{D}}_{\mathcal{X}}^{(\mathfrak{e}+s)}$  such that the image of  $\{\partial_{\langle l \rangle}\}_{l \in \mathbb{N}^d, |l| \leq n}$  in  $\mathcal{D}_{\mathcal{X}/p^M}^{(\mathfrak{e}+s)}$  is contained in  $\mathcal{D}_{\mathcal{X}/p^M, n}^{(\mathfrak{e}+s)}$  and that it is the dual basis of  $\{\tau^{\{l\}}\}_{l \in \mathbb{N}^d, |l| \leq n} \subseteq \mathcal{O}_{\mathcal{D}(1)_{(\mathfrak{e}+s)}^n} / p^M \mathcal{O}_{\mathcal{D}(1)_{(\mathfrak{e}+s)}^n}$  for all  $M \in \mathbb{N}$ . (Here  $\mathcal{D}_{\mathcal{X}/p^M}^{(\mathfrak{e}+s)}$ ,  $\mathcal{D}_{\mathcal{X}/p^M, n}^{(\mathfrak{e}+s)}$  are as in the previous section.) Then, for any  $j \in \mathbb{N}^d$  with  $0 < |j| < p^{s-\lfloor \frac{N}{e} \rfloor}$ ,  $\partial_{\langle j \rangle}((\widetilde{F}^s)^*(x)) = 0$  for any  $x \in \mathcal{E}$ .*

*Proof.* It suffices to prove that, for any  $l \in \mathbb{N}^d$ , the coefficient of  $\tau^j$  in the image of  $\tau^{\{l\}}$  (where  $\{l\}$  is considered with respect to  $\mathfrak{e}$ -PD structure) by the map (2.4) is zero modulo  $\varpi^N$ . It suffices to prove it for  $l$  of the form  $(0, \dots, l_i, \dots, 0)$  ( $l_i > 0$ ) for some  $i$ . In this case, the coefficient of  $\tau^j$  is zero unless  $j$  is of the form  $(0, \dots, j_i, \dots, 0)$ . So we can put  $l := l_i$ ,  $j := j_i$  and regard them as elements in  $\mathbb{N}$ .

We estimate the additive  $\varpi$ -adic valuation  $v$  of the binomial coefficient  $\binom{p^s}{j}$ , assuming  $0 < j < p^{s-\lfloor \frac{N}{e} \rfloor}$ . If we denote by  $\alpha(n)$  ( $n \in \mathbb{N}$ ) the sum of digits of the  $p$ -adic expansion of  $n$ ,  $v$  is given by  $v = \frac{e(\alpha(j) + \alpha(p^s - j) - 1)}{p-1}$ . Under the condition  $0 < j < p^{s-\lfloor \frac{N}{e} \rfloor}$ , we have

$$p^s > p^s - j > p^{s-\lfloor \frac{N}{e} \rfloor} (p^{\lfloor \frac{N}{e} \rfloor - 1} + \dots + 1)(p-1)$$

and so  $\alpha(p^s - j) = (p-1)\lfloor \frac{N}{e} \rfloor + \alpha(p^{s-\lfloor \frac{N}{e} \rfloor} - j)$ . So

$$v = e \left( \left\lfloor \frac{N}{e} \right\rfloor + \frac{\alpha(j) + \alpha(p^{s-\lfloor \frac{N}{e} \rfloor} - j) - 1}{p-1} \right).$$

On the right hand side, the second term inside the bracket is equal to or greater than 1 because it is the  $p$ -adic additive valuation of the binomial coefficient  $\binom{p^{s-\lfloor \frac{N}{e} \rfloor}}{j}$ . So we have the estimate  $v \geq e(\lfloor \frac{N}{e} \rfloor + 1) \geq N$ . From this and the calculation (2.6), we see that the coefficient of  $\tau_i^j$  in the image of  $\tau_i$  by the map (2.4) is zero modulo

$\varpi^N$ , which we denote by  $\varpi^N c_j$ . Then the coefficient of  $\tau_i^j$  in the image of  $\tau_i^{\{l\}}$  by the map (2.4) is equal to

$$\frac{\varpi^{Nl}}{q_l!} \sum_{\substack{j_1 + \dots + j_l = j \\ j_i > 0}} c_{j_1} \cdots c_{j_l},$$

where  $q_l := \lfloor \frac{l}{p^\epsilon} \rfloor$ . We should prove that it is zero modulo  $\varpi^N$ . So it suffices to prove that the additive  $\varpi$ -adic valuation  $w$  of  $\frac{\varpi^{Nl}}{q_l!}$  is equal to or greater than  $N$ , which follows from the calculation

$$w = Nl - \frac{e(q_l - \alpha(q_l))}{p-1} > Nl - \frac{el}{p^\epsilon(p-1)} \geq Nl - l \geq N - 1.$$

So we are done.  $\square$

Next we explain the definition of the level raising Frobenius pullback functor (2.1) in global situation. So let  $X, \mathcal{X}$  be projective again. Let us take an open covering  $\mathcal{X} = \bigcup_\alpha \mathcal{X}_\alpha$  of  $\mathcal{X}$  such that each  $\mathcal{X}_\alpha$  admits a local coordinate and a lift  $\tilde{F}_\alpha^s$  of  $s$ -times iteration of relative Frobenius as in the local situation. Then the definition in local situation says that, for an  $\mathcal{O}_{\mathcal{X}^{[s]}}$ -coherent quasi-nilpotent  $\widehat{\mathcal{D}}_{\mathcal{X}^{[s]}}^{(m)}$ -module  $\mathcal{E}$ ,  $(\tilde{F}_\alpha^s)^* \mathcal{E}$  has a structure of  $\mathcal{O}_{\mathcal{X}_\alpha}$ -coherent quasi-nilpotent  $\widehat{\mathcal{D}}_{\mathcal{X}_\alpha}^{(m+s)}$ -module. So it suffices to glue this local definition. Let us put  $\mathcal{X}_{\alpha\beta} := \mathcal{X}_\alpha \cap \mathcal{X}_\beta$ . Let  $\mathcal{D}(1)_{(m), \alpha\beta}$  be the open formal subscheme of  $\mathcal{D}(1)_{(m)}$  which is homeomorphic to  $\mathcal{X}_{\alpha\beta}$  and let  $p_i : \mathcal{D}(1)_{(m), \alpha\beta} \rightarrow \mathcal{X}_{\alpha\beta}$  ( $i = 1, 2$ ) be projections. Also, let  $^{[s]}$  denote the pullback of formal schemes by  $\pi^s$ . Then it is known that the morphism

$$\tilde{F}_\beta^s|_{\mathcal{X}_{\alpha\beta}} \times \tilde{F}_\alpha^s|_{\mathcal{X}_{\alpha\beta}} : \mathcal{X}_{\alpha\beta} \rightarrow \mathcal{X}_{\alpha\beta}^{[s]} \times_V \mathcal{X}_{\alpha\beta}^{[s]}$$

factors through  $\mathcal{D}(1)_{(m), \alpha\beta}^{[s]}$ . (Here we use the assumption  $m \geq \mathfrak{e}$ .) The structure of  $\widehat{\mathcal{D}}_{\mathcal{X}^{[s]}}^{(m)}$ -module on  $\mathcal{E}$  induces that of an HPD-stratification, which induces an isomorphism  $\epsilon : p_2^* \mathcal{E} \rightarrow p_1^* \mathcal{E}$  on  $\mathcal{D}(1)_{(m), \alpha\beta}^{[s]}$ . By pulling it back to  $\mathcal{X}_{\alpha\beta}$  via the morphism induced by  $\tilde{F}_\beta^s|_{\mathcal{X}_{\alpha\beta}} \times \tilde{F}_\alpha^s|_{\mathcal{X}_{\alpha\beta}}$ , we obtain the isomorphism  $\tilde{F}_\alpha^* \mathcal{E} \rightarrow \tilde{F}_\beta^* \mathcal{E}$ , and we can check that this gives the glueing data. So we obtain the level raising Frobenius pullback  $\mathbb{F}^{s*} \mathcal{E}$ . (In fact, it is known that the isomorphism above is an isomorphism of  $\widehat{\mathcal{D}}_{\mathcal{X}_{\alpha\beta}}^{(m+s)}$ -modules and so  $\mathbb{F}^{s*} \mathcal{E}$  has a structure of  $\widehat{\mathcal{D}}_{\mathcal{X}}^{(m+s)}$ -module.)

Let  $\mathcal{E}$  be a  $V/\varpi^{N+1}V$ -flat, coherent  $\mathcal{O}_{\mathcal{X}^{[s]}}/\varpi^{N+1}\mathcal{O}_{\mathcal{X}^{[s]}}$ -module such that  $\mathcal{E}/\varpi^N \mathcal{E} = (\mathcal{O}_{\mathcal{X}^{[s]}}/\varpi^N \mathcal{O}_{\mathcal{X}^{[s]}})^r$ . Then  $\mathcal{E}$  is a locally free  $\mathcal{O}_{\mathcal{X}^{[s]}}/\varpi^{N+1}\mathcal{O}_{\mathcal{X}^{[s]}}$ -module of rank  $r$ . Take a sufficiently fine open covering  $\mathcal{X}^{[s]} = \bigcup_\alpha \mathcal{X}_\alpha^{[s]}$  of  $\mathcal{X}$  and fix an isomorphism  $\mathcal{E}|_{\mathcal{X}_\alpha^{[s]}} \cong (\mathcal{O}_{\mathcal{X}_\alpha^{[s]}}/\varpi^{N+1}\mathcal{O}_{\mathcal{X}_\alpha^{[s]}})^r$  which lifts the fixed equality  $\mathcal{E}/\varpi^N \mathcal{E} = (\mathcal{O}_{\mathcal{X}^{[s]}}/\varpi^N \mathcal{O}_{\mathcal{X}^{[s]}})^r$ . Then, on each  $\mathcal{X}_{\alpha\beta}^{[s]}$ , we have an isomorphism

$$(\mathcal{O}_{\mathcal{X}_{\alpha\beta}^{[s]}}/\varpi^{N+1}\mathcal{O}_{\mathcal{X}_{\alpha\beta}^{[s]}})^r \cong (\mathcal{E}|_{\mathcal{X}_\alpha^{[s]}})|_{\mathcal{X}_{\alpha\beta}^{[s]}} = (\mathcal{E}|_{\mathcal{X}_\beta^{[s]}})|_{\mathcal{X}_{\alpha\beta}^{[s]}} \cong (\mathcal{O}_{\mathcal{X}_{\alpha\beta}^{[s]}}/\varpi^{N+1}\mathcal{O}_{\mathcal{X}_{\alpha\beta}^{[s]}})^r$$

which lifts the identity on  $(\mathcal{O}_{\mathcal{X}_{\alpha\beta}^{[s]}}/\varpi^N \mathcal{O}_{\mathcal{X}_{\alpha\beta}^{[s]}})^r$ . So this map has the form  $1 + \varpi^N \Lambda_{\alpha\beta}$ , where  $\Lambda_{\alpha\beta} \in \Gamma(X_{\alpha\beta}^{[s]}, M_n(\mathcal{O}_{X^{[s]}}))$ . (Here  $X_{\alpha\beta}^{[s]} := \mathcal{X}_{\alpha\beta}^{[s]} \otimes_V k$ .) Then  $e(\mathcal{E}) := \{\Lambda_{\alpha\beta}\}$  defines an element of  $H^1(X^{[s]}, M_n(\mathcal{O}_{\mathcal{X}^{[s]}})) = H^1(X^{[s]}, \mathcal{O}_{X^{[s]}})^{r^2}$ , and it is trivial if and only if  $\mathcal{E}$  is isomorphic to  $(\mathcal{O}_{\mathcal{X}^{[s]}}/\varpi^{N+1} \mathcal{O}_{\mathcal{X}^{[s]}})^r$ . The next proposition calculates  $e(\mathbb{F}^* \mathcal{E})$  when  $\mathcal{E}$  has a structure of quasi-nilpotent  $\widehat{\mathcal{D}}_{\mathcal{X}^{[s]}}^{(m)}$ -module for sufficiently large  $m$ :

**Proposition 2.7.** *Let  $N \geq 1$ . Let  $\mathcal{E}$  a  $V/\varpi^{N+1}V$ -flat, coherent  $\mathcal{O}_{\mathcal{X}^{[s]}}/\varpi^{N+1} \mathcal{O}_{\mathcal{X}^{[s]}}$ -module endowed with a structure of quasi-nilpotent  $\widehat{\mathcal{D}}_{\mathcal{X}^{[s]}}^{(m)}$ -module such that  $\mathcal{E}/\varpi^N \mathcal{E} = (\mathcal{O}_{\mathcal{X}^{[s]}}/\varpi^N \mathcal{O}_{\mathcal{X}^{[s]}})^r$  as  $\mathcal{O}_{\mathcal{X}^{[s]}}$ -modules, and let  $\mathbb{F}^* \mathcal{E}$  be its image by the level raising Frobenius pullback. Then, if  $m \geq 2N + \epsilon + 1$ ,  $e(\mathbb{F}^* \mathcal{E}) = F^* e(\mathcal{E})$ , where  $F^*$  on the right hand side is the map  $H^1(X^{[s]}, \mathcal{O}_{X^{[s]}})^{r^2} \longrightarrow H^1(X^{[s-1]}, \mathcal{O}_{X^{[s-1]}})^{r^2}$  induced by relative Frobenius.*

*Proof.* Let us take a sufficiently fine open covering  $\mathcal{X}^{[s-1]} = \bigcup_{\alpha} \mathcal{X}_{\alpha}^{[s-1]}$  of  $\mathcal{X}^{[s-1]}$  as above such that  $\mathcal{X}_{\alpha}^{[s-1]}$  admits a local coordinate and a lift of relative Frobenius  $\tilde{F}_{\alpha} : \mathcal{X}_{\alpha}^{[s-1]} \longrightarrow \mathcal{X}_{\alpha}^{[s]}$  as in the local definition of level raising Frobenius pullback. Then we have the isomorphism  $\mathcal{E}|_{\mathcal{X}_{\alpha}^{[s]}} \cong (\mathcal{O}_{\mathcal{X}_{\alpha}^{[s]}}/\varpi^{N+1} \mathcal{O}_{\mathcal{X}_{\alpha}^{[s]}})^r$  as above, and  $\mathbb{F}^* \mathcal{E}$  is defined by glueing  $(\mathcal{O}_{\mathcal{X}_{\alpha}^{[s-1]}}/\varpi^{N+1} \mathcal{O}_{\mathcal{X}_{\alpha}^{[s-1]}})^r \cong \tilde{F}_{\alpha}^* \mathcal{E}$  via the composite

$$(\mathcal{O}_{\mathcal{X}_{\alpha\beta}^{[s-1]}}/\varpi^{N+1} \mathcal{O}_{\mathcal{X}_{\alpha\beta}^{[s-1]}})^r \cong (\tilde{F}_{\alpha}^* \mathcal{E})|_{\mathcal{X}_{\alpha\beta}^{[s-1]}} \cong (\tilde{F}_{\beta}^* \mathcal{E})|_{\mathcal{X}_{\alpha\beta}^{[s-1]}} \cong (\mathcal{O}_{\mathcal{X}_{\alpha\beta}^{[s-1]}}/\varpi^{N+1} \mathcal{O}_{\mathcal{X}_{\alpha\beta}^{[s-1]}})^r,$$

where the isomorphism in the middle comes from the structure of HPD-stratification on  $\mathcal{E}$ . Hence, an element  $\tilde{F}_{\alpha}^*(\mathbf{e}) \in (\mathcal{O}_{\mathcal{X}_{\alpha\beta}^{[s-1]}}/\varpi^{N+1} \mathcal{O}_{\mathcal{X}_{\alpha\beta}^{[s-1]}})^r$  is sent by this isomorphism to

$$(2.7) \quad (1 + \varpi^N F^*(\Lambda_{\alpha\beta})) \left( \sum_{l \in \mathbb{N}^d} (\tilde{F}_{\beta}^*(t) - \tilde{F}_{\alpha}^*(t))^{\{l\}} \tilde{F}_{\alpha}^*(\partial_{\langle l \rangle}(\mathbf{e})) \right),$$

where the local coordinate  $t$  and the differential operators  $\partial_{\langle l \rangle}$  are the ones we took on  $\mathcal{X}_{\alpha}^{[s]}$  and in  $\mathcal{D}_{\mathcal{X}_{\alpha}^{[s]}}^{(m)}$ , respectively.

Since  $\mathcal{E}$  is a  $\varpi^{N+1}$ -torsion  $\mathcal{O}_{\mathcal{X}^{[s]}}$ -coherent quasi-nilpotent  $\widehat{\mathcal{D}}_{\mathcal{X}^{[s]}}^{(m)}$ -module, it is written as the image of a  $\varpi^{N+1}$ -torsion  $\mathcal{O}_{\mathcal{X}^{[s+m-\epsilon]}}$ -coherent quasi-nilpotent  $\widehat{\mathcal{D}}_{\mathcal{X}^{[s+m-\epsilon]}}^{(\epsilon)}$ -module by  $\mathbb{F}^{m-\epsilon*}$ . Hence, by Lemma 2.6,  $\partial_{\langle l \rangle}$  acts on some basis  $B$  of  $\mathcal{E}|_{\mathcal{X}_{\alpha\beta}^{[s]}}$  by zero when  $0 < |l| < p^{m-\epsilon-\lfloor \frac{N+1}{e} \rfloor}$ . So, when  $\mathbf{e}$  runs through the above basis  $B$  of  $\mathcal{E}|_{\mathcal{X}_{\alpha\beta}^{[s]}}$ , the terms which may survive in the big bracket in (2.7) are the constant term  $\tilde{F}_{\alpha}^*(\mathbf{e})$  and the terms with  $(\tilde{F}_{\alpha}^*(t) - \tilde{F}_{\beta}^*(t))^{\{l\}}$ ,  $|l| \geq p^{m-\epsilon-\lfloor \frac{N+1}{e} \rfloor}$ . Because  $\tilde{F}_{\alpha}^*(t) - \tilde{F}_{\beta}^*(t)$  is divisible by  $\varpi$ ,  $(\tilde{F}_{\alpha}^*(t) - \tilde{F}_{\beta}^*(t))^{\{l\}}$  is contained in  $p^c \mathcal{E}$ , where  $c$  is the additive  $\varpi$ -adic valuation of  $\varpi^{|l|}/q_{|l|}!$ , where  $q_{|l|} = \lfloor |l|/p^m \rfloor$ . Then if we denote by  $\alpha(x)$  ( $x \in \mathbb{N}$ ) the sum of digits of the  $p$ -adic expansion of  $x$ , we have the following estimate for  $c$ :

$$c = |l| - \frac{e(q_{|l|} - \alpha(q_{|l|}))}{p-1} > |l| - \frac{e|l|}{p^m(p-1)} \geq p^{m-\epsilon-\lfloor \frac{N+1}{e} \rfloor} \left( 1 - \frac{e}{p^m(p-1)} \right).$$



Then, when  $m \geq 2N + \mathfrak{e} + 1$ ,  $m - \mathfrak{e} - \lfloor \frac{N+1}{e} \rfloor \geq N$  and  $m \geq \mathfrak{e} + 3$ . So

$$p^{m-\mathfrak{e}-\lfloor \frac{N+1}{e} \rfloor} \left( 1 - \frac{e}{p^m(p-1)} \right) \geq 2^N \left( 1 - \frac{1}{8} \right) \geq N.$$

So  $c \geq N + 1$  and (2.7) is equal to  $(1 + \varpi^N F^*(\Lambda_{\alpha\beta}))(\mathbf{e})$  when  $\mathbf{e}$  runs through the basis  $B$  of  $\mathcal{E}|_{\mathcal{X}_{\alpha\beta}^{[s]}}$ . Thus we see that  $e(\mathbb{F}^*\mathcal{E}) = \{F^*(\Lambda_{\alpha\beta})\} = F^*e(\mathcal{E})$ .  $\square$

**Corollary 2.8.** *Let  $\mathcal{E}$  be as in the proposition above with  $s \geq b$ , where  $b$  is as in Proposition 2.4. Then  $\mathbb{F}^{b*}\mathcal{E}$  is isomorphic to  $(\mathcal{O}_{\mathcal{X}^{[s-b]}}/\varpi^{N+1}\mathcal{O}_{\mathcal{X}^{[s-b]}})^r$  as  $\mathcal{O}_{\mathcal{X}^{[s-b]}}$ -modules.*

*Proof.* By Proposition 2.7,  $e(\mathbb{F}^{b*}\mathcal{E}) = F^{b*}e(\mathcal{E})$  and by definition of  $b$  given in the proof of Proposition 2.4, it is zero. So  $F^{b*}\mathcal{E}$  is isomorphic to  $(\mathcal{O}_{\mathcal{X}^{[s-b]}}/\varpi^{N+1}\mathcal{O}_{\mathcal{X}^{[s-b]}})^r$  as  $\mathcal{O}_{\mathcal{X}^{[s-b]}}$ -modules.  $\square$

Now we give a proof of Theorem 1.7.

*Proof of Theorem 1.7.* Let  $r \in \mathbb{N}$  be the rank of  $\mathcal{E}$  and take  $a, b \in \mathbb{N}$  such that the conclusion of Propositions 2.2, 2.4 are satisfied. By making  $a, b$  larger, we may assume that  $a \geq b \geq \mathfrak{e} + 3$ .

First, note that, for any  $m' \leq m$  and for any  $i$ , a quasi-nilpotent  $\widehat{\mathcal{D}}_{\mathcal{X}^{[i]}}^{(m)}$ -module can be regarded also as a quasi-nilpotent  $\widehat{\mathcal{D}}_{\mathcal{X}^{[i]}}^{(m')}$ -module. Using this, we can replace the infinite set of  $m \in \mathbb{N}$  for which  $\mathcal{G}^{[i](m)}$  is defined by any infinite subset of  $\mathbb{N}$ . So we put  $I := \{m \mid b|(m - a - \mathfrak{e}), m \geq a + \mathfrak{e}\}$  and assume in the following that  $\mathcal{G}^{[i](m)}$  is defined for any  $m \in I$ . Now take any  $m = b(m' - 1) + a + \mathfrak{e} \in I$ . For  $0 \leq j \leq m - \mathfrak{e}$ , let  $\mathcal{G}^{[i](m)[j]}$  be the  $j$ -th Frobenius antecedent of  $\mathcal{G}^{[i](m)}$  and put  $G^{[i](m)[j]} := \mathcal{G}^{[i](m)[j]}/\varpi \mathcal{G}^{[i](m)[j]}$ . Since  $\mathcal{G}^{[i](m)[j]}$  is  $p$ -torsion free,  $\mu(G^{[i](m)[j]}) = \mu(\mathbf{E}^{[i+j]}) = 0$ ,  $p_{\mathcal{G}^{[i](m)[j]}} = p_{\mathbf{E}^{[i+j]}} = p_{\mathcal{O}_X}$ . Also,  $G^{[i](m)[j]}$  is  $\mu$ -stable (hence stable): Indeed, for any coherent subsheaf  $0 \neq H \subsetneq G^{[i](m)[j]}$ ,  $p^j \mu(H) = \mu(F^{j*}H) < \mu(G^{[i](m)}) = 0$ . (Here the inequality follows from the  $\mu$ -stability of  $G^{[i](m)}$ .) Hence any subsequence of length  $a$  of the sequence  $\{G^{[i](m)[j]}\}_{j=0}^{m-\mathfrak{e}}$  satisfies the assumption of Proposition 2.2. Hence the sequence  $\{G^{[i](m)[j]}\}_{j=0}^{b(m'-1)}$  is the constant sequence  $\{\mathcal{O}_{X^{[i+j]}}\}_{j=0}^{j=b(m'-1)}$ . (So  $r$  should be 1.)

In particular,  $G^{[i](m)[b(m'-1)]} = \mathcal{G}^{[i](m)[b(m'-1)]}/\varpi \mathcal{G}^{[i](m)[b(m'-1)]}$  is isomorphic to  $\mathcal{O}_X = \mathcal{O}_{\mathcal{X}}/\varpi \mathcal{O}_{\mathcal{X}}$  as  $\mathcal{O}_{\mathcal{X}}$ -modules. Because it has a structure of  $\widehat{\mathcal{D}}_{\mathcal{X}}^{(a)}$ -module, it has a structure of  $\widehat{\mathcal{D}}_{\mathcal{X}}^{(b)}$ -module. We prove by induction that  $\mathcal{G}^{[i](m)[b(m'-l)]}/\varpi^l \mathcal{G}^{[i](m)[b(m'-l)]}$  is isomorphic to  $\mathcal{O}_{\mathcal{X}^{[i+b(m'-1)]}}/\varpi^l \mathcal{O}_{\mathcal{X}^{[i+b(m'-1)]}}$  as  $\mathcal{O}_{\mathcal{X}^{[i+b(m'-1)]}}$ -modules and it has a structure of  $\widehat{\mathcal{D}}_{\mathcal{X}^{[i+b(m'-1)]}}^{(lb)}$ -module. Indeed, assume that this is true for  $l$ . Then, since  $lb \geq l(\mathfrak{e} + 3) \geq 2l + \mathfrak{e} + 1$ ,

$$\mathbb{F}^{b*}(\mathcal{G}^{[i](m)[b(m'-l)]}/\varpi^{l+1} \mathcal{G}^{[i](m)[b(m'-l)]}) = \mathcal{G}^{[i](m)[b(m'-l-1)]}/\varpi^{l+1} \mathcal{G}^{[i](m)[b(m'-l-1)]}$$

is isomorphic to  $\mathcal{O}_{\mathcal{X}^{[i+b(m'-l-1)]}}/\varpi^{l+1} \mathcal{O}_{\mathcal{X}^{[i+b(m'-l-1)]}}$  as  $\mathcal{O}_{\mathcal{X}^{[i+b(m'-l-1)]}}$ -modules by Corollary 2.8, and it has a structure of  $\widehat{\mathcal{D}}_{\mathcal{X}^{[i+b(m'-l-1)]}}^{((l+1)b)}$ -modules. So we see finally that  $\mathcal{G}^{[i](m)}/\varpi^{m'} \mathcal{G}^{[i](m)}$  is isomorphic to  $\mathcal{O}_{\mathcal{X}^{[i]}}/\varpi^{m'} \mathcal{O}_{\mathcal{X}^{[i]}}$  as  $\mathcal{O}_{\mathcal{X}^{[i]}}$ -modules.

Now let us move  $m$ : Put  $i_0 := \min_{m \in I} (i(m))$  and take  $m_0 \in I$  with  $i_0 = i(m_0)$ . First we check that  $\mathbf{E}^{[i_0]}$  is  $\mu$ -stable. Let  $\mathfrak{G}$  be the coherent  $\mathcal{O}_{\mathfrak{X}}$ -module corresponding to  $\mathcal{G}^{[i_0](m_0)}$  via GFGA. Then, for any coherent subsheaf  $0 \neq \mathbf{E}' \subsetneq \mathbf{E}^{[i_0]}$ ,  $\mathfrak{G}' := \mathfrak{G} \cap (\alpha_* \mathbf{E}')$  (where  $\alpha : \mathbf{X} \rightarrow \mathfrak{X}$  is the canonical open immersion) is a  $p$ -torsion free coherent subsheaf of  $\mathfrak{G}$  such that  $\mathfrak{G}/\mathfrak{G}'$  is again  $p$ -torsion free. Hence  $\mathfrak{G}'/\varpi \mathfrak{G}'$  is a coherent subsheaf of  $\mathfrak{G}/\varpi \mathfrak{G} = G^{[i_0](m_0)}$  and so  $\mu(\mathbf{E}') = \mu(\mathfrak{G}'/\varpi \mathfrak{G}') < \mu(G^{[i_0](m_0)}) = \mu(\mathbf{E}^{[i_0]})$  because  $G^{[i_0](m_0)}$  is  $\mu$ -stable.

Next, for  $m = b(m'-1) + a + \mathfrak{e} \in I$ , let  $\mathcal{H}^{(m)}$  be the level raising Frobenius pullback of  $\mathcal{G}^{[i](m)}$  by  $(i - i_0)$ -times iteration of relative Frobenius. Then  $\mathcal{H}^{(m)}/\varpi^{m'} \mathcal{H}^{(m)}$  is isomorphic to  $\mathcal{O}_{\mathcal{X}^{[i_0]}}/\varpi^{m'} \mathcal{O}_{\mathcal{X}^{[i_0]}}$  and so  $\mathcal{H}^{(m)}/\varpi \mathcal{H}^{(m)}$  is also  $\mu$ -stable for any  $m \in I$ .

Hence, by theorem of Langton [11],  $\mathcal{H}^{(m)}$ 's have the form  $\varpi^{c_m} \mathcal{H}^{(m_0)}$  for some  $c_m \in \mathbb{Z}$  depending on  $m$ . This implies that  $\mathcal{H}^{(m_0)}/\varpi^{m'} \mathcal{H}^{(m_0)}$  is isomorphic to  $\mathcal{H}^{(m)}/\varpi^{m'} \mathcal{H}^{(m)}$ , thus to  $\mathcal{O}_{\mathcal{X}^{[i_0]}}/\varpi^{m'} \mathcal{O}_{\mathcal{X}^{[i_0]}}$  for all  $m \in I$ . Therefore,  $\mathcal{H}^{(m_0)}$  is isomorphic to  $\mathcal{O}_{\mathcal{X}^{[i_0]}}$ . So  $\mathcal{E}^{[i_0]} = \mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{H}^{(m_0)}$  is isomorphic to  $\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{X}^{[i_0]}}$ . To show the triviality of the action of  $\mathcal{D}_{\mathcal{X}^{[i_0]}, \mathbb{Q}}^\dagger$  on  $\mathcal{E}^{[i_0]}$ , it suffices to see that  $H^0(\mathcal{X}^{[i_0]}, \mathbb{Q} \otimes_{\mathbb{Z}} \Omega_{\mathcal{X}^{[i_0]}}^1) = H^0(\mathbf{X}^{[i_0]}, \Omega_{\mathbf{X}^{[i_0]}}^1) = 0$ . This follows from the (in)equalities

$$\dim H^0(\mathbf{X}^{[i_0]}, \Omega_{\mathbf{X}^{[i_0]}}^1) \leq \dim H_{\text{dR}}^1(\mathbf{X}^{[i_0]}) = \dim H_{\text{et}}^1(\overline{\mathbf{X}}^{[i_0]}, \mathbb{Q}_l) = \dim H_{\text{et}}^1(X^{[i_0]}, \mathbb{Q}_l) = 0.$$

Hence  $\mathcal{E}^{[i_0]}$  is a trivial convergent isocrystal and so is  $\mathcal{E}$ .  $\square$

Next we give a proof of Theorem 1.9.

*Proof of Theorem 1.9.* Let  $r, a, b$  as in the proof of 1.7.

First, by the same reason as before, we can replace the infinite set of  $m \in \mathbb{N}$  for which  $\mathcal{G}^{[i](m)}$  is defined by any infinite subset of  $\mathbb{N}$ . Also, we can replace  $l(m) (\leq m - \mathfrak{e})$  for each  $m$  by any smaller value, keeping the property  $l(m) \rightarrow \infty (m \rightarrow \infty)$ . Using this, we see that we may take an infinite subset  $I'$  of  $\mathbb{N}$ , the set  $I := \{c \mid br|(c - ar), c \geq br(r - 1) + ar\}$  and a bijection  $l : I' \rightarrow I$  so that  $\mathcal{G}^{[i](m)}$  is defined for any  $m \in I'$  and that  $l = l(m)$  in the statement of Theorem 1.9 is given by the image of  $m$  by the map  $l$  above.

Now take any  $m \in I'$  so that  $l := l(m) = br(m' + r - 2) + ar$ . For  $0 \leq j \leq l$ , define the Jordan-Hölder filtration [9, Definition 1.5.1]  $\{U_q^{[i](m)[j]}\}_{q=0}^{q_j}$  of  $G^{[i](m)[j]}$  in the following way, by descending induction: First, when  $j = l$ , take any Jordan-Hölder filtration  $\{U_q^{[i](m)[l]}\}_{q=0}^{q_l}$  of  $G^{[i](m)[l]}$ . When we defined  $\{U_q^{[i](m)[j+1]}\}_{q=0}^{q_{j+1}}$ , the pull-back  $\{F^* U_q^{[i](m)[j+1]}\}_{q=0}^{q_{j+1}}$  of it by relative Frobenius defines a filtration of  $F^* G^{[i](m)[j+1]} = G^{[i](m)[j]}$  whose graded pieces are semistable. Then we define  $\{U_q^{[i](m)[j]}\}_{q=0}^{q_j}$  by taking any Jordan-Hölder filtration which refines  $\{F^* U_q^{[i](m)[j+1]}\}_{q=0}^{q_{j+1}}$ . By definition, we have

$$r \geq q_0 \geq q_1 \geq \cdots \geq q_l \geq 1,$$

where  $r$  is the rank of  $\mathcal{E}$ . So, there exists some  $j_0$  such that  $q_{j_0} = \cdots = q_{j_0 + b(m' + r - 2) + a} (= : Q)$ .

Put  $V_q^{[i](m)[j]} := U_q^{[i](m)[j]} / U_{q-1}^{[i](m)[j]}$ . Then, for each  $1 \leq q \leq Q$ , any subsequence of length  $a$  of the sequence  $\{V_q^{[i](m)[j]}\}_{j=j_0}^{j=j_0+b(m'+r-2)+a}$  satisfies the assumption of Proposition 2.2. Hence  $\{V_q^{[i](m)[j]}\}_{j=j_0}^{j=j_0+b(m'+r-1)}$  is isomorphic to the constant sequence  $\{\mathcal{O}_{X^{[i+j]}}\}_{j=j_0}^{j_0+b(m'+r-2)}$ . Then, we can apply Proposition 2.4 to any subsequence of length  $b(r-1)$  of the sequence  $\{G^{[i](m)[j]}\}_{j_0}^{j=j_0+b(m'+r-2)}$ . So  $\{G^{[i](m)[j]}\}_{j=j_0}^{j_0+b(m'-1)}$  is isomorphic to the constant sequence  $\{\mathcal{O}_{X^{[i+j]}}^r\}_{j=j_0}^{j_0+b(m'-1)}$ . Then, by the same argument as the proof of Theorem 1.7, we see from Proposition 2.7 that  $\mathcal{G}^{[i](m)} / \varpi^{m'} \mathcal{G}^{[i](m)}$  is isomorphic to  $(\mathcal{O}_{X^{[i]}} / \varpi^{m'} \mathcal{O}_{X^{[i]}})^r$ .

Now let us move  $m$ : Put  $i_0 := \min_{m \in I'}(i(m))$ , and for each  $m \in I'$  with  $l(m) = br(m' + r - 2) + ar \in I$ , let  $\mathcal{H}^{(m)}$  be the level raising Frobenius pullback of  $\mathcal{G}^{[i](m)}$  by  $(i - i_0)$ -times iteration of relative Frobenius. Note that  $\mathcal{H}^{(m)} / \varpi^{m'} \mathcal{H}^{(m)}$  is isomorphic to  $(\mathcal{O}_{X^{[i_0]}} / \varpi^{m'} \mathcal{O}_{X^{[i_0]}})^r$  and so  $\mathcal{H}^{(m)} / \varpi \mathcal{H}^{(m)}$  is semistable. On the other hand,  $\mathbf{E}^{[i_0]}$  is also semistable. (This can be proven in the same way as the proof of Theorem 1.7.)

Let  $\overline{M}$  be the moduli of semistable sheaves on  $\mathfrak{X}^{[i_0]}$  with rank  $r$  and reduced Hilbert polynomial  $p_{\mathcal{O}_X}$ , which is constructed by Adrian Langer ([1], [2]). It is a projective scheme over  $\text{Spec } V$ . Then, for any  $m \in I$ ,  $\mathcal{H}^{(m)}$  defines a  $V$ -valued point  $P_m$  of  $\overline{M}$  which induces the  $K$ -valued point  $P_K$  defined by  $\mathbf{E}^{[i_0]}$ . Since  $\overline{M}$  is separated, the  $V$ -point which extends  $P_K$  is unique. Hence  $P_m$  is independent of  $m$ , which we denote by  $P$ . (This does not imply that  $\mathcal{H}^{(m)}$  are independent of  $m$  because  $\overline{M}$  is not a fine moduli.) On the other hand, let  $P'$  be the  $V$ -valued point defined by  $\mathcal{O}_{\mathfrak{X}^{[i_0]}}^r$ . Since  $\mathcal{H}^{(m)} / \varpi^{m'} \mathcal{H}^{(m)}$  is trivial, the  $V / \varpi^{m'} V$ -valued point induced by  $P$  is the same as the  $V / \varpi^{m'} V$ -valued point induced by  $P'$ . So  $P = P'$  and this implies that  $P_K$  is equal to the  $K$ -valued point defined by  $\mathcal{O}_{\mathbf{X}^{[i_0]}}^r$ . Hence  $\mathbf{E}^{[i_0]}$  is  $S$ -equivalent [9, Definition 1.5.3] to  $\mathcal{O}_{\mathbf{X}^{[i_0]}}^r$  when pulled back to the geometric fiber  $\overline{\mathbf{X}}^{[i_0]}$ , namely,  $\mathbf{E}^{[i_0]}|_{\overline{\mathbf{X}}^{[i_0]}}$  is an iterated extension of  $\mathcal{O}_{\overline{\mathbf{X}}^{[i_0]}}$ . Since  $H^1(\overline{\mathbf{X}}^{[i_0]}, \mathcal{O}_{\overline{\mathbf{X}}^{[i_0]}}) \subseteq H_{\text{dr}}^1(\overline{\mathbf{X}}^{[i_0]}) = 0$ ,  $\mathbf{E}^{[i_0]}|_{\overline{\mathbf{X}}^{[i_0]}}$  is isomorphic to  $\mathcal{O}_{\overline{\mathbf{X}}^{[i_0]}}^r$ , and this implies that  $\mathbf{E}^{[i_0]}$  is isomorphic to  $\mathcal{O}_{\mathbf{X}^{[i_0]}}^r$ . Then, by the same method as the proof of Theorem 1.7, we see that  $\mathcal{E}^{[i_0]}$  is a trivial convergent isocrystal and so is  $\mathcal{E}$ .  $\square$

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